

## On Asymptotic Distribution of Sample Variance In Skew Normal Distribution

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**Abstract:** The univariate skew normal distribution was introduced by Azzalini(1985). The properties of the resulting distribution were studied. The main aim of the paper is to find the limiting distribution of sample variance in skew normal distribution. In this way, the theory of U-statistics is applied.

**Key words:** Mmoments, Limiting distribution, Skew normal distribution, Symmetric Kernel, U-Statistics.

### INTRODUCTION

The univariate skew normal distribution has been considered by several authors. Azzalini (1985) introduced this distribution. A random variable  $X$  has skew normal distribution if

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi(\lambda x), \quad x \in R$$

where  $\Phi$  is cumulative distribution function of standard normal distribution. The skew normal distribution allows for continuous variation from normality to non-normality, which is useful in many practical situations (Hill and Dixon, 1982, Arnold et al., 1983). Nadarajah, et al., (2003) generated

skewed probability density function of the form  $2f(u)G(\lambda u)$  where  $f$  is taken to be a normal pdf while the cumulative distributive function  $G$  is taken to come from one of normal, Student's  $t$ , Cauchy, Laplace, logistic or uniform distribution. In particular, expressions for the  $n$ th moment and characteristic function were derived. In skew normal distribution, the  $n$ th moment of  $X$  about zero turns out to be

$$E(X^n) = \frac{2^{\frac{n+2}{2}} \lambda \Gamma(\frac{n+2}{2})}{\pi} (1 + \lambda^2)^{-\frac{n}{2}} \sum_{k=0}^{\frac{n-1}{2}} \frac{\binom{1-n}{2}_k}{\left(\frac{3}{2}\right)_k} (-\lambda^2)^k, \tag{1}$$

if  $n$  is odd, and

$$E(X^n) = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right), \tag{2}$$

for  $n$  even, where  $\binom{c}{k} = c(c+1)\cdots(c+k-1)$

In this paper, by using these results, we derive the asymptotic distribution of sample variance in skew normal distribution.

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**RESULT AND DISCUSSION**

Let  $X_1, X_2, \dots, X_n$  be a random sample from skew normal distribution. The U-statistic estimator of the variance with the symmetric kernel  $h(X_1, X_2) = \frac{1}{2}(X_1 - X_2)^2$  is

$$\begin{aligned}
 U_2(X_1, \dots, X_n) &= \frac{1}{\binom{n}{2}} \sum_{\beta \in B} \frac{1}{2} \left[ \sum_{i=1}^2 X_{\beta_i}^2 - 2X_{\beta_1} X_{\beta_2} \right] \\
 &= \frac{1}{n(n-1)} \left[ n \sum_{i=1}^n X_i^2 - \left( \sum_{i=1}^n X_i \right)^2 \right] \\
 &= S^2,
 \end{aligned}$$

where  $B = \{ \beta \mid \beta \text{ is one of the } \binom{n}{2} \text{ unordered subsets of 2 integers chosen without replacement from the set } \{1, 2, \dots, n\} \}$  (Koroljik and Borovskich 1994). Since the conditions of theorem 3 from Ferguson (2005),

$(\mu_4 = E(X - \mu)^4 \sigma^4 < \mu_4 < \infty, E_{\mathcal{F}} h^2 < \infty)$  holds, we conclude that the sample variance has a

limiting normal distribution with mean  $\sigma^2$  and variance  $\frac{\mu_4 - \sigma^4}{n}$

We will employ (1) and (2) to establish the limiting distribution of  $S^2$ .

$$\begin{aligned}
 E(X) &= \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{1}{\sqrt{1 + \lambda^2}}, & E(X^2) &= 1, \\
 E(X^3) &= \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{3 + 2\lambda^2}{(1 + \lambda^2)^{\frac{3}{2}}}, & E(X^4) &= 3.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \mu_4 - \sigma^4 &= E(X^4) - 4E(X^3)E(X) + 8E(X^2)E^2(X) - E^2(X^2) - 4E^4(X) \\
 &= 3 - 4 \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{3+2\lambda^2}{(1+\lambda^2)^{\frac{3}{2}}} \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{1}{\sqrt{1+\lambda^2}} + 8 \left( \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{1}{\sqrt{1+\lambda^2}} \right)^2 \\
 &\quad - 1 - 4 \left( \frac{\sqrt{2}}{\sqrt{\pi}} \lambda \frac{1}{\sqrt{1+\lambda^2}} \right)^4 \\
 &= 2 - \frac{8\lambda^2}{\pi} \frac{3+2\lambda^2}{(1+\lambda^2)^2} + \frac{16\lambda^2}{\pi} \frac{1}{(1+\lambda^2)} - \frac{16\lambda^2}{\pi^2} \frac{1}{(1+\lambda^2)^2} \\
 &= 2 - \frac{8\lambda^2}{\pi^2 (1+\lambda^2)^2} [3\pi + 2\pi\lambda^2 - 2\pi - 2\pi\lambda^2 + 2\lambda^2] \\
 &= 2 - \frac{8\lambda^2 (\pi + 2\lambda^2)}{\pi^2 (1+\lambda^2)^2}.
 \end{aligned}$$

So,  $\sqrt{n}(S^2 - 1 + \frac{2\lambda^2}{\pi(1+\lambda^2)}) \sim AN(0, 2 - \frac{8\lambda^2 (\pi + 2\lambda^2)}{\pi^2 (1+\lambda^2)^2})$ . If  $\lambda$  tends to infinity the above value

near to  $2 - \frac{16}{\pi^2}$  and

$$\sqrt{n}(S^2 - 1 + \frac{2}{\pi}) \sim AN(0, 2 - \frac{16}{\pi^2}).$$

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