Application of Variational Iteration Method to the Estimation of Electric Potential in 2D Plate with Infinite Length

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Abstract: Many electrical components depend on potential parameter, so estimation of electric potential is one of the most significant issues. In this work, a powerful analytical method, called variational iteration method (VIM) is investigated to analyze the electrical potential in 2D plate with infinite length. Comparison of these two methods to finite element method (FEM) makes article hotter.

Key words: electric potential, 2D plate with infinite length, variational iteration method (VIM), finite element method (FEM)

INTRODUCTION

The solution of electromagnetic field problems in 2D plate with infinite length obtained by Partial differential equations which arise in real-world physical problems are often too complicated to be solved exactly. And even if an exact solution is obtainable, will be required calculations that may be too complicated to be practical. Using Maxwell's equations electric potential and electric field are determined as solution of Poisson's equation in plate, while in that case Maxwell's equations are non-linear partial differential equations, which solution depends on the initial conditions and earlier media situation. These equations and existing boundary conditions on the separating surface electric scalar potential functions are determined as the solution of Laplace's equations in plate many different new methods have been presented recently such as the variational iteration method (VIM) by (He, 1999; He, 1998; He, 1998; He, 2000; He, 2006; He and Wu, 2006). VIM has many merits over classical approximate techniques which can solve nonlinear equations easily and accurately. This method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a functional and cause a rapid convergent sequence is produced. The variational iteration method is suitable for finding the approximation of the solution without discretization of the problem (He, 2000). and has recently been applied to various engineering problems (Tatari and Dehghan, 2006; Bildik and Konuralp, 2006; Momani and Abuasa, 2006; Odiabat and Momani, 2006; Mehdi Dehghan, Soliman and 2006; Mahdi et al., 2007; Mehdi Tatari, 2007; Shaher Momani, 2007; Mehdi Tatari). In this Letter, we analyzed variational iteration method to estimate electric potential in 2D plate with infinite length by applying this powerful modern numerical approach to Laplace's equations with predetermined boundary condition. Figure (1) shows outlook of 2D plate with infinite length and its condition. Comparison of this new method to final element method (FEM) shows excellent agreement of these two methods. The rest of this paper is organized as follows: section 2 explains some relationship that governs stationary electricity and reach to necessity of powerful approach to estimate electric parameter. Section 3 describes in detail the proposed method. Sections 4 will illustrate and analyze method in various boundary conditions. Section 5 shows the simulation results. Finally, conclusions are presented in Section 6.

Electric potential in differential form:

Maxwell's equations are partial differential forms that are true at all space. One of these main equations is (2-1) that governs stationary electricity.

\[ \nabla \cdot \mathbf{D} = \rho \]  \hspace{1cm} (2-1)

\[ \nabla \times \mathbf{E} = 0 \]  \hspace{1cm} (2-2)
Fig. 1: Outlook of 2D plate with infinite length.

Where
\[ \nabla \text{ is Del operator, } D \text{ is electric field density, } \rho \text{ is free volume charge density and } E \text{ is electric field.} \]

Nature of non-eddy E as shown in (2-2) makes we have:

\[ E = \nabla \varepsilon \]  

(2-3)

At isotropic environment, \( D = \varepsilon E \) nt, so (2-1) introduced as:

\[ \nabla \varepsilon \varepsilon = \rho \]  

(2-4)

Where
\[ \varepsilon \] is environment permittivity factor. Replacing (3-2) in (3-3):

\[ \nabla (\varepsilon \nabla \varepsilon) = -\rho \]  

(2-5)

In homogeneous environment \( \varepsilon \) is constant; so (2-5) change to:

\[ \nabla \varepsilon = -\frac{\rho}{\varepsilon} \]  

(2-6)

Where
\[ \nabla^2 \] is laplacian operator; (2-6) is well known as Poisson's equation that is partial differential equation. In Cartesian coordinates:

\[ \nabla^2 \varepsilon = \nabla \cdot \nabla \varepsilon = \frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial y^2} + \frac{\partial^2 \varepsilon}{\partial z^2} \]  

(2-7)

Finally replacing (2-7) with (2-6):

\[ \frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial y^2} + \frac{\partial^2 \varepsilon}{\partial z^2} = -\frac{\rho}{\varepsilon} \]  

(2-8)

Solution of Poisson's equation usually is not easy. In environment with no free charge \( \rho = 0 \), (2-8) can be simplified as:

\[ \nabla^2 \varepsilon = 0 \]  

(2-9)

So Presence of powerful analytical approach to solve this problem seems to be critical.

**Basic idea of variational iteration method:**

To clarify the basic ideas of He's VIM, we consider the following differential equation:

\[ Lu + Nu = g(x,t) \]  

(3-1)

Where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(t) \) an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda (Lu_n(\xi) + Nu_n(\xi) - g(\xi)) d\xi \quad n \geq 0 \]  

(3-2)

Where
is a general Lagrangian multiplier (He, 1999; He, 1998; He, 1998; He, 2000; He and Wu, 2006; He, 2006), which can be identified optimally via the variational theory (He, 2006). The subscript indicates the $n$th approximation, and is considered as a restricted variation (He, 2006), i.e. $\lambda = 0$. Therefore, we first determine the Lagrange multiplier $\lambda$ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x, t) \geq 0$ of the solution $u(x, t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $U_0$. The zeroth approximation $U_0$ may be selected by any function that justifies at least two of the prescribed boundary conditions. With $\lambda$ determined, then several approximations $u_{n+1}(x, t), n \geq 0$, follow immediately. Consequently, the exact solution may be obtained by using:

### Applications:
In order to illustrate the method discussed above, we apply VIM to estimate potential in plate which is infinite in Z-axis. It means that in Eq. (2-8):

$$\frac{\partial V}{\partial z} \rightarrow 0$$

There is no free charge in space, $\rho = 0$. The Eq. (2-8) simplified as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{4-1}$$

$\nu_1, \nu_2, \nu_3, \nu_4$ are defined as initial boundary condition as it shown in figure (1).

### Case study:
The Laplace equation is:

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0 \tag{4-2}$$

With this boundary condition:

$$V(0, y) = \sin y \quad y < \pi \tag{4-3}$$
$$V(x, 0) = 0 \quad x > 0$$
$$V(\pi, y) = \cosh \pi \sin y$$
$$V(x, \pi) = 0$$

The correction functional for this equation reads

$$V_{n+1}(x, y) = V_n(x, y) + \int_0^\pi \lambda(\xi) \left( \frac{\partial^2 V_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 V_n(\xi, y)}{\partial y^2} \right) d\xi \tag{4-4}$$

This yields the stationary conditions

$1 - \lambda \xi |_{\xi = x} = 0$ \hspace{1cm} (4-5)
$$\lambda |_{\xi = x} = 0$$
$$\lambda^* |_{\xi = x} = 0$$

This in turn gives

$$\lambda = \xi - x \tag{4-6}$$

Substituting this value of the Lagrange multiplier into the functional Eq. (4-4) gives the iteration formula

$$V_{n+1}(x, y) = V_n(x, y) + \int_0^\pi (\xi - x) \left( \frac{\partial^2 V_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 V_n(\xi, y)}{\partial y^2} \right) d\xi \tag{4-7}$$
The best zeroth approximation that satisfies three of the boundary conditions is $v(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, substituting this value into Eq. (4-7) we obtain the following successive approximations:

\begin{align*}
V_0(x,y) &= \left(1 + \frac{1}{2!} x^2\right) \sin y, \\
V_1(x,y) &= \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4\right) \sin y, \\
V_2(x,y) &= \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6\right) \sin y, \\
V_3(x,y) &= \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \ldots\right) \sin y,
\end{align*}

\(4-8\)

Recall that

\begin{equation}
V(x,y) = \cosh x \sin y \tag{4-9}
\end{equation}

Obtained upon using the Taylor expansion for $\cosh x$.

**Case study:**

Boundary conditions of Laplace equation in this example are:

\begin{align*}
V(0, y) &= 0 \quad y < \pi \\
V(\pi, y) &= \sinh \pi \cdot \cos y \\
V(x, 0) &= \sinh x \quad x > 0 \\
V(x, \pi) &= -\sinh x
\end{align*}

\(4-10\)

The correction functional for this equation reads

\begin{equation}
V_{n+1}(x, y) = V_n(x, y) + \int_0^\pi \lambda(\xi) \left(\frac{\partial^2 V_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 V_n(\xi, y)}{\partial y^2}\right) d\xi \tag{4-11}
\end{equation}

This in turn gives

\begin{equation}
\lambda = \xi - x \tag{4-12}
\end{equation}

Substituting this value of the Lagrange multiplier into the functional Eq. (4-11) gives the iteration formula:

\begin{equation}
V_{n+1}(x, y) = V_n(x, y) + \int_0^\pi (\xi - x) \left(\frac{\partial^2 V_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 V_n(\xi, y)}{\partial y^2}\right) d\xi \tag{4-13}
\end{equation}

Considering the given boundary conditions, it is clear that the solution contains $\cos y$ in addition to other functions that depend on $x$. Therefore, we can select $V_0(x,y)$. The zeroth approximation $V_0(x,y)$ satisfies three boundary conditions when considering $\sinh x = x$. Using this selection into Eq. (4-13) we obtain the following successive approximations:

\begin{equation}
V_n(x,y) = x \cos y
\end{equation}
This gives the exact solution by
\[ V(x, y) = \sinh x \cos y \] (4-15)

Obtained upon using the Taylor expansion for
The Taylor expansion for \( \sinh x \).

**Case study:**

Boundary conditions of Laplace equation in this example are:
\[ V_x(0, y) = 0 \]
\[ V_x(\pi, 0) = 0 \]
\[ V_x(x, 0) = \cos x \quad x > 0 \]
\[ V_x(x, \pi) = \cosh \pi \cos x \]

The boundary conditions are Neumann boundary conditions where the derivatives of the solution are
\( u(x, y) \) specified at the edges. It is normal to select Using
\[ V_0(x, y) = \left( x + \frac{1}{3!} x^3 \right) \cos x \]
\[ V_1(x, y) = \left( x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 \right) \cos x \]
\[ V_2(x, y) = \left( x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + ... \right) \cos x \]

Selecting \( V_n(x, y) \) that satisfies three of the boundary conditions, gives the following
successive approximations:
\[ V_0(x, y) = \left( x + \frac{1}{3!} y^3 \right) \cos x \]
\[ V_1(x, y) = \left( x + \frac{1}{3!} y^3 + \frac{1}{5!} y^5 \right) \cos x \]
\[ V_2(x, y) = \left( x + \frac{1}{3!} y^3 + \frac{1}{5!} y^5 + \frac{1}{7!} y^7 + ... \right) \cos x \] (4-18)

Because the boundary conditions are Neumann boundary conditions, an additive constant \( C \) must be added.

Based on this, the exact solution is given by:
This gives the exact solution by
\[ V(x, y) = C + \sinh y \cos x \] (4-19)
Where \( C \) is a constant.
Simulation by 2D FEM:

In order to analyze the electric potential, 2D Finite Element method was used for the purpose of the FEM to get the overall figure of the electric potential amounts in various points of the 2D plate with infinite length, to compare the electric potential obtained from FEM and analytic method (VIM). It should be mentioned that in this article, magneto-static analysis of PDE TOOLBOX software of MATLAB7 was used for programming and simulation of finite element method (FEM) and all limits were considered.

Finite Element Method:

The FEM is a numerical technique for obtaining approximation solution to boundary value problems of mathematical physics. Especially it has become a very important tool solve electrostatic problems because of its ability to model geometrically and compositionally complex problems.

The potential distribution which satisfies the differential equation, subject to proper boundary conditions, will also minimize the stored energy in the field and vice versa. Therefore one practical approach for solving the field problem is to approximate and minimize the stored energy in the field. To construct and approximate solution by finite element method, the complicated field region is discretized into a number of uniform or ununiform finite elements that are connected via nodes. The potential within each element is approximated by an interpolation function. Thereafter the potential distribution in the various elements is element boundaries. The total energy is the sum of the individual element energies. Then, the total stored energy is minimized. The result of this minimization can be reformulated into a matrix equation:

\[
\begin{align*}
S \cdot A &= N
\end{align*}
\]

In this equation, \( S \) is the complex global matrix whose coefficients are functions of the geometry of region considered, material properties, boundary conditions and angular frequencies. \( N \) is the current vector. The nonlinear matrix equation can be interatively solved to get the potential distribution in the field (Gholamian et al., 2006).

Using FEM to solve problems involves three stages. The consists of meshing the problem space into contiguous elements of the suitable geometry and assigning appropriate values of the material parameter – conductivity, permeability and permittivity – to each element. Secondary, the model has to be excited, so that the initial conditions are set up. Finally, the values of the potentials are suitably constrained at the limits of the problem space. The finite element method has the advantage of geometrical flexibility. It is possible to include a greater density of elements in regions where fields and geometry vary rapidly (Gholamian et al., 2006).

Simulation of Case study 1 using FEM:

Calculation of the electric potential distribution in the 2D plate with infinite length is presented in this Section. For simulation, the model is first divided into 10000 meshes.

The three-dimensional figure of the electric potential distribution for our defined model is obtained, as shown in Fig.2.

Fig. 2: Three-dimensional of the electric potential using VIM

Fig.3 shows comparison between the FEM results and VIM analysis of electric potential distribution over y axis at in \( x = \frac{d}{2} \).
Fig. 3: Comparison between the FEM and VIM results

Fig. 4 shows comparison three-dimensional of the electric potential distribution between the FEM and VIM over defined model.

Fig. 4: Comparison between the FEM and VIM in Case study 1.

**Simulation of Case study 2 using FEM:**
Calculation of the electric potential distribution in Case study 2 is presented in this Section. The three-dimensional figure of the electric potential distribution for our defined model in Case study 2 is obtained, as shown in Fig. 5.

Fig. 5: Three-dimensional of the electric Potential in Case study 2 using VIM.

Fig. 6 shows 2D plot comparison between the FEM results and VIM analysis of electric potential distribution over x axis in $y = \frac{b}{8}$.
Fig. 6: Comparison between the FEM and VIM results.

Fig. 7 shows comparison three-dimensional of the electric potential distribution between the FEM and VIM over in Case study 2.

Fig. 7: Comparison between the FEM and VIM in Case study 2.

**Simulation of Case study 3 using FEM:**

Calculation of the electric potential distribution in Case study 3 is presented in this Section. The three-dimensional figure of the electric potential distribution for our defined model in Case study 2 is obtained, as shown in Fig. 8.

Fig. 8: Three-dimensional of the electric Potential in Case study 3 using VIM.

Fig. 9 shows 2D plot comparison between the FEM results and VIM analysis of electric potential distribution over x axis in \( y = \frac{b}{2} \).
Fig. 9: Comparison between the FEM and VIM results.

Fig. 10 shows comparison three-dimensional of the electric potential distribution between the FEM and VIM over in Case study 3.

Conclusions:
In this Letter we have estimated electric potential in 2D plate with infinite length with the Variational iteration method. This method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a functional and cause a rapid convergent sequence is produced. Finally, it has been attempted to show the capabilities and facile applications of the Variational iteration method in comparison with the finite element method (FEM). Results of Electric potential comparison between the various points of the defined model of 2D plate with infinite length show that the FEM and VIM analysis are in excellent agreement.

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