Numerical Solution of Linear Fredholm Fuzzy Integral Equation of the Second Kind by Block-pulse Functions

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Abstract: In this paper, we present a numerical method based on block-pulse functions (BPFs) for the solution of a system of linear Fredholm integral equation of second kind with two variable. We will apply our method for an applied example, linear Fredholm fuzzy integral equation of the second kind (M. Friedman, M. Ming, A. Kandel, Numerical solution of fuzzy differential and integral equation, fuzzy Sets and Systems 106 (1999) 35-48).

Key words: block-pulse functions; Fredholm fuzzy integral equation. MSC: 65R20, 45A05, 45B05, 41A30

INTRODUCTION

In recent years, many basic functions have used to estimate the solution of system of integral equation, such as orthogonal bases Wavelets, for example, see (Maleknadj et al., 2007). In this paper, we are going to use a kind of these bases, a simple base, that is a block-pulse functions on [0, 1).

BPFs are studied by many authors and applied for solving different problems, for example, see (Maleknejad et al., 2005; Babolian and Masouri, 2008).

Block-pulse Functions:

An m-set of block-pulse functions (BPFs) is defined over the interval [0, T) as:

\[
\phi_i(t) = \begin{cases} 
\frac{iT}{m}, & \frac{iT}{m} \leq t < \frac{(i+1)T}{m} \\
0, & \text{otherwise}
\end{cases}
\]

where \(i = 0, 1, \ldots, m - 1\) with a positive integer value for \(m\). Also, consider \(h = T/m\), and \(\phi_i\) is the \(i\)th Block-Pulse function.

In this paper, it is assumed that \(T = 1\), so BPFs is defined over [0, 1), and \(h = 1/m\) there are some properties for BPFs, the most important properties are disjointness, orthogonality. The disjointness property can be clearly obtained from the definition of BPFs:

\[
\phi_i(t)\phi_j(t) = \begin{cases} 
\phi_i(t), & i = j \\
0, & i \neq j
\end{cases}
\]

where \(i, j = 0, 1, \ldots, m - 1\).

The other property is orthogonality. It is clear that:

\[
\int_0^1 \phi_i(t)\phi_j(t)\,dt = h\delta_{ij}
\]
where $\delta_{ij}$ is the Kroneker delta. The third property is completeness. For every $f \in L^2([0, 1])$ when $m$ approaches to the infinity, Parseval’s identity holds:

$$\int_0^1 f^2(t) \, dt = \sum_{i=0}^{m} f_i^2 \| \phi_i(t) \|^2,$$

where,

$$f_i = \frac{1}{m} \int_0^1 f(t) \phi_i(t) \, dt$$

**Vector Forms:**

consider the first $m$ terms of BPFs, and write them concisely as $m$-vector:

$$\Phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_m(t)]^T, t \in [0, 1) (6)$$

above representation and disjointness property, follows:

$$\Phi(t) \Phi^T(t) = F \Phi^T(t) \Phi(t)$$

where $F$ is diagonal matrix $m \times m$ with diagonal elements equal to entries of $m$-vector $F$.

**Fuzzy Numbers:**

**Definition 1:**

A fuzzy number is a pair of functions satisfying following requirements:

(i) $\bar{u}(r)$ is a bounded monotonic increasing left continuous function.

(ii) $\bar{v}(r)$ is a bounded monotonic decreasing left continuous function.

(iii) $\bar{u}(r) \leq \bar{v}(r), 0 \leq r \leq 1$.

This definition is giving by Kaleva (). The set of all fuzzy numbers is denoted by $F$. for arbitrary $u = (u, \bar{u}), v = (v, \bar{v})$, and $k > 0$ define $(u + v)$ and multiplication by $k$ as

$$\overline{u + v}(r) = \underline{u}(r) + \underline{v}(r)$$

$$\overline{u + v}(r) = \underline{u}(r) + \underline{v}(r)$$

Definition 2:

for arbitrary fuzzy number $u = (u, \bar{u}), v = (v, \bar{v})$ the quantity no

$$D(u, v) = \max \left( \sup_{0 \leq r \leq 1} [u(r) - v(r)], \sup_{0 \leq r \leq 1} [\bar{u}(r) - \bar{v}(r)] \right)$$

is the distance between $u$ and $v$.

let $f : [a, b] \to E_1$, for each partition $p = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ with $h = \max |t_i - t_{i-1}|$ and for arbitrary
The definite integral of \( f(t) \) over \([a, b]\) is
\[
\int_a^b f(t)\,dt = \lim_{\Delta \to 0} \sum_{i=1}^n f(x_i)(t_i - t_{i-1}).
\]
(12)

provided that this limit exists in the metric \( D \). If the fuzzy function \( f(t) \) is continuous in the metric \( D \), its definite integral exists. Furthermore,
\[
\int_a^b f(t, r)\,dt = \int_a^b f(t, r)\,dr.
\]
(13)

the integral equations which are discussed in this section are the Fredholm equations of second kind. The Fredholm integral equation of the second kind is (FFIE-2) see (Dubois and Prade, 1982).
\[
u(t) = f(t) + \beta \int_a^b K(s, t)\nu(s)\,ds.
\]
(13)

where \( \beta > 0 \), \( K(s, t) \) is an arbitrary Kernel function over the square \( a \leq s, t \leq b \) and \( f(t) \) is a function of \( t : a \leq t \leq b \). If \( f(t) \) is a fuzzy function these equations may only possess fuzzy solutions. Now, we introduce parametric form of a FFIE-2 with respect to Definition 1. Let \( (\tilde{f}(t, r), \overline{f}(t, r)) \) are parametric form of \( f(t) \) and \( u(t) \), respectively, then parametric form of FFIE-2 is as follows:
\[
u(t, r) = f(t, r) + \beta \int_a^b \nu_1(s, t\,u(s, r), \overline{u}(s, r))\,ds.
\]
\[
u(t, r) = f(t, r) + \beta \int_a^b \nu_2(s, t\,u(s, r), \overline{u}(s, r))\,ds.
\]
(14)

Where
\[
\nu_1(s, t\,u(s, r), \overline{u}(s, r)) = \begin{cases} K(s, t)\nu(s, r) & K(s, t) \geq 0, \\ K(s, t)\overline{u}(s, r) & K(s, t) < 0, \end{cases}
\]

and
\[
\nu_2(s, t\,u(s, r), \overline{u}(s, r)) = \begin{cases} K(s, t)\nu(s, r) & K(s, t) \geq 0, \\ K(s, t)\overline{u}(s, r) & K(s, t) < 0, \end{cases}
\]

for each \( 0 \leq r < 1 \) and \( a \leq t \leq b \). We can see that (14) is a linear system of Fredholm integral equations with two variables in crisp case for each \( 0 \leq r < 1 \) and \( a \leq t \leq b \).

**BPF expansion:**

the expansion of a function \( f(t) \) over \([0, 1]\) with respect to \( \phi(t) \), \( i = 0, 1, \ldots, m-1 \) may be compactly written as:
\[
f(t) = \sum_{i=1}^{m-1} f_i \phi(i) - P^T \Phi(t) - \Phi^T(t) P,
\]
(15)
where \( P = [f_1, f_2, \ldots, f_{m_2}] \) and \( f_i \)'s defined by (4) now, assume \( h(t, s) \) is a function of two variables in \( L([0, 1] \times [0, 1]) \). It can be similarly expanded with respect to BPFs such as:

\[
h(t, s) = \Phi^T(t) H \Psi(s)
\]

(16)

where \( \Phi(t) \) and \( \Psi(s) \) are \( m_1 \), \( m_2 \) dimensional BPF vectors respectively, and \( H \) is the \( m_1 \times m_2 \) block-pulse coefficient matrix with \( i = 0, 1, \ldots, m_1 - 1, j = 0, 1, \ldots, m_2 - 1 \) as follows:

\[
h_{ij} = m_1 m_2 \int_0^1 \int_0^1 h(t, s) \phi_i(t) \psi_j(s) dt ds
\]

(17)

for convenience, we put \( m_1 = m_2 = m \). We also define the matrix \( D \) as follows:

\[
D = \int_0^1 \Phi(t) \Phi^T(t)
\]

(18)

\( D \) has the following form:

\[
D = \begin{pmatrix}
  h & 0 & \cdots & 0 \\
  0 & h & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & h
\end{pmatrix}
\]

**System of Linear Integral Equation with Two Variables:**

consider the following integral equation system:

\[
\sum_{j=1}^{m_2} f_{ij}(t) y_j(t, r) + \sum_{j=1}^{m_2} k_{ij}(t, s) y_j(s, r) ds = x_i(t, r), \quad i = 1, 2, \ldots, n,
\]

(19)

where \( f_{ij}, x_i, y_j, k_{ij} \in L([0, 1] \times [0, 1]) \) for \( i, j = 1, 2, \ldots, n \) and \( y_j \) are unknown function, if we approximate \( f_{ij}, x_i, x_i(t, s) = \Phi(t)^T X_i \Phi(r), \quad f_{ij} = \Phi^T(t) \Phi(t), \quad k_{ij}(t, s) = \Phi(t)^T K_{ij} \Phi(s), \quad y_j(t, r) = \Phi(t)^T Y_j \Phi(r) \).

(20)

With substituting in (19) we have

\[
\sum_{j=1}^{m_2} \Phi(t)^T \tilde{F}_{ij} \Phi(r) + \sum_{j=1}^{m_2} \Phi(t)^T K_{ij} \Phi(s) \Phi(s)^T Y_j \Phi(r) ds = \Phi(t)^T X_i \Phi(r),
\]

\[i = 1, 2, \ldots, n.
\]

\[
\Rightarrow \sum_{j=1}^{m_2} \Phi(t)^T \tilde{F}_{ij} \Phi(r) + \sum_{j=1}^{m_2} \Phi(t)^T D K_{ij} Y_j \Phi(r) = \Phi(t)^T X_i \Phi(r),
\]

\[i = 1, 2, \ldots, n.
\]

\[
\Rightarrow \sum_{j=1}^{m_2} (\tilde{F}_{ij} + D K_{ij}) Y_j = X_i, \quad i = 1, 2, \ldots, n
\]

set \( \Gamma_j = \tilde{F}_{ij} + D K_{ij} \) then we have linear system:

\[
\begin{pmatrix}
  \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1n} \\
  \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \Gamma_{n1} & \Gamma_{n2} & \cdots & \Gamma_{nn}
\end{pmatrix}
\begin{pmatrix}
  Y_1 \\
  Y_2 \\
  \vdots \\
  Y_n
\end{pmatrix}
\begin{pmatrix}
  X_1 \\
  X_2 \\
  \vdots \\
  X_n
\end{pmatrix}
\]

(20)
by solving this linear system we can find matrices $Y_j, j=1, 2, \ldots, n$ so: 
$y_j(t,r) = \Phi(t)^T Y_j \Phi(r)$

**Experiments:**

Now we apply these methods for two examples, we compare approximate solutions with exact solutions (see Table 1 and 2). The computations associated with the examples were performed using Matlab 7.

**Example 1:**

(Friedman et al., 1999). Consider the FFIE-2 with

$$\bar{f}(t, r) = -\frac{1}{3} \pi \sin(\pi t)(-20 + 5r^3 + 7r + 2r^2),$$

$$\bar{f}(t, r) = \frac{1}{3} \pi \sin(\pi t)(5r^2 + 7r - 8 + 2r^3),$$

and kernel

$$K(t, r) = \pi \sin(2\pi s) \sin(\pi t), 0 \leq s, t \leq 1$$

and $a = 0, b = 1$. The exact solution in this case is given by

$u(t, r) = (r^4 + r) \pi \sin(\pi t)$

$\bar{u}(t, r) = (4 - r^3 - r) - \pi \sin(\pi t)$

**Example 2:**

Consider the following FFIE-2 with:

$$\bar{f}(t, r) = -\frac{1}{3} \pi \sin(\pi t)(-20 + 5r^3 + 7r + 2r^2),$$

$$\bar{f}(t, r) = \frac{1}{3} \pi \sin(\pi t)(5r^2 + 7r - 8 + 2r^3),$$

and kernel

$$K(s, t) = (2t - 1)^2(1 - 2s), 0 \leq s, t \leq 1$$

and $a = 0, b = 1$. The exact solution in this case is given by

$u(t, r) = (2 - r) t$

$\bar{u}(t, r) = rt$.

**Table 1:** Numerical result for Example 1 with $m=64, \tau=0.6$

<table>
<thead>
<tr>
<th>$r$</th>
<th>Exact solution $u$</th>
<th>Approximate solution $u$</th>
<th>Exact solution $u$</th>
<th>Approximate solution $u$</th>
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**Table 2:** Numerical result for Example 2 with $m=32, \tau=0.5$

<table>
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<tr>
<th>$\tau$</th>
<th>Exact solution $u$</th>
<th>Approximate solution $u$</th>
<th>Exact solution $u$</th>
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</table>

also sadeghi goghary (Hossin et al., 2006) has solving system integral equation(19) by an expansion method. Comparing this method with the our method, shows that the number of calculations of the our method presented in this artical is lower.

**Conclusion:**

In this work, we present a numerical method based on block-pulse functions (BPFs). We can see that solving the system of the linear Fredholm integral equations of second kind with two variables is converted to solving some systems of linear equations. By two examples we tried to find an approximate solution for the results seem acceptable.

The benefits of this method are low cost of setting up the equations without applying any projection method such as Galerkin, collocation, etc. Also, the number of calculations is very low.

**REFERENCES**


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