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Magnetohydrodynamic Stability of Self-Gravitating Fluid Layer Embedded into Different Fluid

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Abstract: The magnetohydrodynamic stability of self-gravitating fluid layer embedded into an infinite fluid of different density has been developed. Such a model has crucial applications in several domains of science. The problem is formulated, the partial differential equations are solved, the boundary conditions are applied and the dispersion relation is derived. The stability criterion established for all kinds of the perturbations. The self-gravitating force is destabilizing as the net dimensionless wave number is less than 0.64 while it is stabilizing for all the rest. The electromagnetic force has strong stabilizing influence and that influence increases and suppressed the self gravitating destabilizing effect and stabilizing sets in.

Key words:

INTRODUCTION

The stability of a semi-infinite layer under natural forces has been discussed by Rayleigh (1945). Chandrasekhar (1981) did discuss the stability of two semi-infinite fluids instability under the fluid pressure gradient and the external gravitating forces as the fluids are initially at rest (Rayleigh-Taylor) or not (Kelvin-Helmholtz instability). Soon, after wards Chandrasekhar (1981) made several extensions for such studies and others. See also Kant and Malik (1985), Oganesyan (1961), Radwan (1987), (1990) and (2007) and Craik (1966). The latter author laid the theoretical foundation for the shear perturbation stability, see also Benjamin (1959) and Miles (1962). The response of the self-gravitating instability of a fluid cylinder for axisymmetric perturbation is being studied for first time by Chandrasekhar and Fermi (1953). Later on Chandrasekhar (1981) (Nobile prize winner 1986) developed and made miscellaneous extensions. In every chapter of Chandrasekhar's book (1981) there has been no attempt to give exhaustive references nor does the author claim to have read or even found every relevant paper.

Here we study the magnetohydrodynamic stability of self-gravitating layer embedded into different fluid, for all modes of perturbations.

Basic State:

We have a fluid layer of thickness $2h$ embedded into a different infinite fluid. The latter fluid of density ρ^e and the fluid layer of density ρ^i are assumed to be incompressible-non-viscous and perfectly conducting and pervaded by the magnetic fields

$$\underline{H}_o^{i,e} = (H_o, 0, 0) \tag{1}$$

The fluids are acted upon by the pressure gradient, self-gravitating and magnetohydrodynamic forces.

The basic equations which are required for discussing the stability of such kind of problems are the combination of the equations concerning the electromagnetic theory, ordinary hydrodynamic equations and the Newtonian self-gravitating equations. These equations are given as follows

$$\rho^{(s)}\left(\frac{\partial}{\partial t} + u^{(s)}\square\nabla\right)\underline{u}^{(s)} = \mu^{(s)}\left(\nabla \wedge \underline{H}^{(s)}\right) \wedge \underline{H}^{(s)} - \rho^{(s)}\nabla V^{(s)} - \nabla P^{(s)} \tag{2}$$

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$$\nabla \square \underline{u}^{(s)} = 0 \qquad \nabla \square \underline{H}^{(s)} = 0 \tag{3,4}$$

$$\left(\frac{\partial}{\partial t} + \underline{u}^{(s)} \square \nabla\right) \underline{H}^{(s)} = \left(\underline{H}^{(s)} \square \nabla\right) \underline{u}^{(s)} - \underline{H}^{(s)} \left(div \underline{u}^{(s)}\right) + \underline{u}^{(s)} \left(div \underline{H}^{(s)}\right) \tag{5}$$

$$\nabla^2 V^{(s)} = 4\pi G \rho^{(s)} \qquad , \qquad s = i, e \tag{6}$$

Equations (2) are the magnetogravitational vector equations of motion, equations (3) are the continuity equations, equations (4) are the conservation of flux, equations (5) are the evaluation equations of magnetic fields and equations (6) are Newtonian's self-gravitating equations. Here $\rho, \underline{u}, P$ are the fluid mass density, velocity vector and pressure, μ and $\underline{H}$ are the magnetic field permeability and intensity and V and G are the self-gravitating potential and constant, respectively.

Perturbation Technique:

For small departures from the unperturbed state, every physical quantity $Q(x, y, z; t)$ could be expressed as its value in the unperturbed $Q_o(z)$ besides an infinitesimal increment $Q_1(x, y, z; t)$, viz.

$$Q(x, y, z; t) = Q_o(z) + \varepsilon(t) Q_1(x, y, z) \tag{7}$$

Here $Q(x, y, z; t)$ stands for $\underline{u}, P, \underline{H}, V$ and the z-distance of the fluid layer, with $\varepsilon(t)$ is, the amplitude of perturbation at instant time t, given by

$$\varepsilon = \varepsilon_o \exp(i\sigma t) \tag{8}$$

where $\varepsilon_o \left(= \varepsilon \text{ at } t = 0\right)$ is the initial amplitude and σ is the oscillation frequency. Due to the perturbation, the z-distance of the perturbed wave along the fluid layer-infinite fluid interface is being

$$z = z_o + z_1 \qquad , \qquad z_o = \pm h \tag{9}$$

with

$$z_1 = \varepsilon_o \exp[i(kx + ly + \sigma t)] \tag{10}$$

is the elevation of the surface wave measured from the unperturbed position where k and l are the wavenumbers in the x and y directions.

From the view point of the expansions (7) - (10), the magnetogravitational linearized equations describing the perturbed state of the fluid layer are given by

$$\rho^i \left(\frac{\partial \underline{u}_1^i}{\partial t} + \left(\underline{u}_o^i \square \nabla\right) \underline{u}_1^i\right) = -\nabla P^i + \mu^i \left(\text{curl}\, \underline{H}_1^i\right) \wedge \underline{H}_o^i \tag{11}$$

$$\nabla \square \underline{u}_1^i = 0 \qquad \nabla \square \underline{H}_1^i = 0 \tag{12,13}$$

$$\left(\frac{\partial}{\partial t} + \underline{u}_o^i \square \nabla\right) \underline{H}_1^i = \left(\underline{H}_o^i \square \nabla\right) \underline{u}_1^i \qquad , \qquad \underline{u}_o = 0 \tag{14}$$

$$\nabla^2 V_1^i = 0 \tag{15}$$

Since we use here the normal mode analysis, based on the linear perturbation technique, we have to seek a solution of perturbation equations whose $(x, y, t) - \text{dependence}$ in the form

$$Q_1(x, y, z; t) = Q_1^*(z)\, e^{i(kx + ly + \sigma t)} \tag{16}$$

Consequently, equation (14) yields

$$\sigma \underline{H}_1 = k H_o \underline{u}_1 \tag{17}$$

Taking the circulation of equation (11), we obtain

$$\rho^i \frac{\partial}{\partial t}\left(\operatorname{curl} \underline{u}_1\right) = -\operatorname{curl}\left(\operatorname{grad} P_1\right) + \mu^i \operatorname{curl}\left(\left(\operatorname{curl} \underline{H}_1^i\right) \wedge \underline{H}_o^i\right) \tag{18}$$

upon using the identities

$$\begin{aligned}\nabla \wedge \left(\left(\nabla \wedge \underline{H}_1\right) \wedge \underline{H}_o\right) = \left(\underline{H}_o \square \nabla\right) \operatorname{curl} \underline{H}_1 - \left(\left(\nabla \wedge \underline{H}_1\right) \square \nabla\right) \underline{H}_o \\ - \left(\nabla \square \left(\nabla \wedge \underline{H}_1\right)\right) \underline{H}_o + \left(\nabla \square \underline{H}_o\right)\left(\nabla \wedge \underline{H}_1\right)\end{aligned} \tag{19}$$

$$\nabla \wedge \left(\nabla P_1\right) = 0 \qquad , \qquad \underline{\omega}_1 = \nabla \wedge \underline{u}_1 \tag{20}$$

equation (18), reduces to

$$A \underline{\omega}_1 = 0 \qquad , \qquad A = \left(i/\sigma\right)\left(\sigma^2 - \left(\mu^i k^2 H_o^2\right)/\rho\right) \tag{21}$$

and so

$$\frac{\partial u_{1y}}{\partial x} - \frac{\partial u_{1x}}{\partial y} = 0 \quad , \quad \frac{\partial u_{1x}}{\partial z} - \frac{\partial u_{1z}}{\partial x} = 0 \quad , \quad \frac{\partial u_{1y}}{\partial x} - \frac{\partial u_{1x}}{\partial y} = 0 \tag{22}$$

Combining equation (22) with the continuity equation (12):

$$iku_{1x} + ilu_{1y} = -\frac{\partial u_{1z}}{\partial z} \tag{23}$$

we get

$$\frac{d^2 u_{1x}(z)}{dz^2} - \left(l^2 + k^2\right) u_{1x}(z) = 0 \tag{24}$$

The non-singular solution of equation (24), is given by

$$u_{1x} = c_1 z_1 \cosh mz \qquad , \qquad m^2 = l^2 + k^2 \tag{25}$$

and consequently

$$u_{1y} = c_1 \left(l/k\right) z_1 \cosh mz \tag{26}$$

$$u_{1z} = -\left(im/k\right) c_1 \sinh mz \tag{27}$$

and

$$\underline{H}_1^i = \left(kH_o/\sigma\right) \underline{u}_1^i \tag{28}$$

Also, the non-singular solution of equation (15) is given by

$$V_1^i = c_2 \cosh mz \tag{29}$$

where c_1 and c_2 are constants of integration to be identified.
For the fluid of density ρ^e surrounding the fluid layer the perturbation equations are given by equations (11)--(15) but with superscript e instead of i. The non-singular solutions of this system of perturbed equations, as $z \to \infty$, are given by

$$u_{1x}^e = c_3 e^{i(\sigma t + kx + ly)} e^{-mz} \tag{30}$$

$$u_{1y}^e = \left(l/k\right) u_{1x}^e \tag{31}$$

$$u_{1z}^e = -\left(im/k\right) u_{1x}^e \tag{32}$$

$$V_1^e = c_4 e^{i(\sigma t + kx + ly)} e^{-mz} \tag{33}$$

$$\underline{H}_1^e = \left(kH_o/\sigma\right) \underline{u}_1^e \tag{34}$$

where c_3 and c_4 are constants of integration to be determined.
In order to obtain the pressures of the fluid layer and the surrounding fluid, let us return to the system of equations (11)--(15). Equation (11) yields

$$\frac{\partial \underline{u}_1^i}{\partial t} - \frac{i\mu k H_o}{\rho^i} \underline{H}_1^i = -\nabla \Pi_1^i \tag{35}$$

with

$$\Pi_1^i = \frac{1}{\rho}\left(P_1^i + \mu\left(\underline{H}_o \square \underline{H}_1^i\right) + \rho^i V_1^i\right) \tag{36}$$

where $\rho\left(\Pi_1 - V_1^i\right) = P_1^i + \mu\left(\underline{H}_o \square \underline{H}_1^i\right)$ is the total MHD pressure which is the sum of the kinetic and magnetic pressures. The z-component of equation (35), is being

$$u_{1z} = \frac{i\sigma}{\left(\sigma^2 - \Omega_A^2\right)} \frac{\partial \Pi_1}{\partial z} \tag{37}$$

from which, we obtain

$$\Pi_1^i = \frac{-\left(\sigma^2 - \left(\Omega_A^i\right)^2\right)}{\sigma k} c_1 \cosh mz \tag{38}$$

where

$$\Omega_A^i = \frac{\mu H_o^2 k}{\rho^i} \tag{39}$$

is the Alfven wave frequency defined in terms of H_o.
In similar steps Π_1^e, for the fluid of density ρ^e surrounding the fluid layer, could be obtained in the form

$$\Pi_1^e = -\frac{1}{\sigma k}\left(\sigma^2 - \left(\Omega_A^e\right)^2\right) c_3\, e^{-mz} \tag{40}$$

with

$$\Pi_1^e = -\frac{1}{\rho^e}\left(P_1^e + \mu^e\left(\underline{H}_o \square \underline{H}_1^e\right) + \rho^e V_1^e\right) \tag{41}$$

As we have seen in the pervious details we get all the variables describing the perturbation state of the fluid layer and the surrounding fluid of density ρ^e. The constants of integrations c_1, c_2, c_3 and c_4 appear and associated with these variables could be determined by using the boundary conditions available to the present model.

(I)Boundary Conditions:

The solution of the basic equations (11)--(15) for the fluid layer (25)--(29) and (37)--(39) and for the surrounding fluid given by (30)--(34) and (40)--(41) must satisfy the following boundary conditions.

(i) Self-gravitating conditions
The self-gravitating potential V and its derivative must be continuous across the fluids interface (9) at $z = h$. These conditions read

$$\left.\begin{aligned} V_1^i + z_1 \frac{\partial V_o^i}{\partial z} &= V_1^e + z_1 \frac{\partial V_o^e}{\partial z} \\ \frac{\partial V_1^i}{\partial z} + z_1 \frac{\partial^2 V_o^i}{\partial z^2} &= \frac{\partial V_1^e}{\partial z} + z_1 \frac{\partial^2 V_o^e}{\partial z^2} \end{aligned}\right\} \quad at\ z = h \tag{42, 43}$$

from which we obtain

$$c_3 = \frac{4\pi G}{m}\left(\rho^e - \rho^i\right)\exp\left(-mh\right) \tag{44}$$

and

$$c_3 = \frac{4\pi G}{m}\left(\rho^e - \rho^i\right)\cosh mh \tag{45}$$

(II) Kinematic Conditions:

The normal component of the fluid layer velocity must be continuous with the velocity of the surrounding fluid and at the same time must be cope with the velocity of the perturbed fluids interface (9) at $z = h$. These conditions give

$$u_{1z}^i = u_{1z}^e = \frac{\partial z}{\partial t} \qquad at \;\; z = h \tag{46}$$

from which, we get

$$c_1 = \frac{-\sigma k}{m \sinh mh} \tag{47}$$

and

$$c_3 = \frac{\sigma k}{m}\;\exp\left(mh\right) \tag{48}$$

(III) Magnetodynamic Condition:

The normal component of the magnetic field must be continuous across the fluid layer-surrounding fluid interface (9) at $z = h$. This condition reads

$$H_{1z}^i = H_{1z}^e \qquad at \;\; z = h_o \tag{49}$$

or alternatively

$$H_o u_{1z}^i = H_o u_{1z}^e \tag{50}$$

By using (27), (32) with (30) and the results (47) and (48) for the condition (50), we see that the condition (49) is identically satisfied at $z = h$.

Finally, we have to apply some compatibility condition across the interface (9) at $z = h$. This condition states that the normal component of the total stress in the fluid layer must be compatible with that of the surrounding fluid across the interface (9) at $z = h$. This condition reads

$$P_1^i + \mu^i\left(\underline{H}_o \cdot \underline{H}_1^i\right) = P_1^e + \mu^e\left(\underline{H}_o \cdot \underline{H}_1^e\right) \tag{51}$$

which may be rewritten, in view of (36) and (41), in the form

$$\rho^i \Pi_1^i - \rho^i V_1^i = \rho^e \Pi_1^e - \rho^e V_1^e \tag{52}$$

By substituting for V_1^i, V_1^e, Π_1^i and Π_1^e, the following dispersion relation is obtained

$$\sigma^2 = \frac{1}{\left(\rho - \coth mh\right)}\left[\left(\rho\frac{\mu^e H_o^2 k^2}{\rho^e} - \frac{\mu^i H_o^2 k^2}{\rho^i}\coth mh\right) - 4\pi G\left(\rho - 1\right)\left(\coth mh\right)\exp\left(-mh\right)\right] \tag{53}$$

where

$$\rho = \rho^e / \rho^i \qquad , \qquad m^2 = l^2 + k^2 \tag{54}$$

DISCUSSIONS

Equation (53) is desired dispersion relation of magnetogravitational fluid layer embedded into magnetogravitational infinite fluid, where both of the layer and surrounding fluid are acted by the fluid pressure gradient, self-gravitating and magnetodynamic forces. It is valid for all kinds of perturbations. Equation (53) relates the oscillation frequency σ with the densities ρ^i of the fluid layer and ρ^e of the surrounding fluid and their ratio $\rho\left(\rho^i/\rho^e\right)$, the wave numbers k and l in x-and y-directions and also the resultant wave number $m\left(=\left(k^2+l^2\right)^{\frac{1}{2}}\right)$ and with the parameters μ^i, μ^e and H_o of the problem.

The analytical investigation and verified numerically of the relation (53) as $\rho^e = 0, H_o = 0$, shows that the fluid layer is self-gravitating unstable as the dimensionless wave number $mh < 0.64$ while it is stable for all waves with wave number $mh > 0.64$ where the marginal stability states occurred at $mh = 0.64$.

As $\rho^e = 0$ and $G = 0$, the discussions of the relation (53) indicate that the electromagnetic force has strong stabilizing effect. The latter effect increases the self-gravitating stable states and simultaneously decreases those of instability and then stability sets in.

In discussing the relation (53) in its general form, we see that the effects of the pressure gradient and electromagnetic forces have modified the stability criterion and improved the instability of the present model.

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