m-Sectorial Operators Associated with The Sesquilinear Forms $t[u,v] = (Tu, v)^*$

Ali Sameripour

Research supported by Lorestan University, Khorramabad, Iran

Abstract: Let $h$ be a densely defined, closed and symmetric form that is bounded from below, and let $H = T_h$ be the associated self-adjoint operator, the relationship $h[u, v] = (Hu, v)$ that connecting the form $h$ with the operator $H$ is unsatisfactory indeed it is not valid for all $u, v \in D(h)$ because $D(H)$ is in general a proper subset of $D(h)$. In this paper, we will study the relation between the m-sectorial operators and the sesquilinear forms $t[u, v] = (Tu, v)$. Moreover, here we will discuss an interesting concepts of sectorial operators; m-sectorial operators and the concept of Friedrichs extension along with an important example.

Key words: resolvent, asymptotic spectrum, eigenvalues, non-self adjoint Elliptic Differential operators

INTRODUCTION

We consider the sesquilinear form $t$ defined on a subspace $L$ of a separable Hilbert space $H$. The sesquilinear form $t[u, v]$ ($u \in L, v \in L$) is given such that:

$t: L \times L \rightarrow \mathbb{C}$.

The form $t[u, v]$ is said to be a sesquilinear form if it satisfies the following conditions:

$t[\alpha u, \beta v] = \alpha \beta t[u, v]$ ($\alpha, \beta \in \mathbb{C}, u, v \in L$)

$t[u_1, u_2, v] = t[u_1, v] + t[u_2, v]$ ($u_1, u_2, v \in L$)

$t[u_1 + u_2, v] = t[u_1, v] + t[u_2, v]$ ($u_1, u_2, v \in L$)

Thus $t[u, v]$ is complex valued and linear in $u \in L$ for each fixed $v \in L$ and semi-linear in $v \in L$ for each fixed $u \in L$. Here $L$ will be called the domain of $t$ and is denoted by $D(t)$ and is densely defined in $H$ i.e., $D(t)$ is dense in $H$. A form $t$ is said to be symmetric if

$t[u, v] = t[v, u]$ ($u, v \in D(t))$.

With each form $t$ associate another form $t^*$ which is defined by:

$t^*[u, v] = \overline{t[v, u]}$ ($D(t) = D^*(t)$)

$t^*$ is called the adjoint of the form $t$, the form $t$ is symmetric if and only if $t = t^*$. Let us now consider a non-symmetric form $t$. The set of values of the form $t$:

$t[u] = \{t[u, v] : u \in D(t) = L : \|u\| = 1\}$

is called the numerical range of $t$ and will be denoted by $\Theta(t)$, the form $t$ is said to be sectorial if $\Theta(t)$ is a subset of a sector of the form.
Here $\gamma$ and $\theta$ will be called a vertex and a semi-angle of the form $t$ respectively.

We denote the inner product and the norm of $H$ by $(\cdot, \cdot)$ and $\|\cdot\|$. If $t(u, v)$ is a sectorial form then there exist numbers $m$ and $\delta > 0$ such that for each $u \in D(t)$ the following inequality satisfies

$$\|t[u, v]\| \leq M \Re t(u, v) + \delta \|u\|^\frac{1}{2}.$$ 

We define the following norm in the space $D[t]$

$$\|u\| = (\Re t[u, v] + \frac{\delta}{M} \|u\|^\frac{1}{2})^\frac{1}{2}.$$ \hspace{1cm} (1.1)

If in such norm the space $D[t]$ is complete (i.e. to be a Banach space) then the sectorial form $t[u, v]$ is called closed form. The following theorem is one of the basic theorems of the theory of m-sectorial forms;

An operator $T$ in $H$ is said to be accretive if the numerical range of $T$ which will be denoted by $\Theta(T)$ is a subset of the right half-plane, that is if

$$\Re \Theta(T) = \{\Re \Theta(t) \in \mathbb{R} : \Re \Theta(t) > 0\}.$$ 

Then $T$ is said to be m-accretive if for $Re \lambda > 0$, $||T + \lambda I|| \leq (Re \lambda)^\frac{1}{2}$ the operator $T$ is said to be quasi-accretive operator if the numerical range $\Theta(T)$, is not only a subset of the right half-plane, but it must be a subset of a sector of the form

$$\mathcal{S} = \{\zeta \in \mathbb{C} : \arg(\zeta - \gamma) \leq \theta \}, \quad 0 \leq \theta < \frac{\pi}{2}, \gamma \in \mathbb{R}.$$ 

We define the following norm in the space $D[t]$

$$\|u\| = (\Re t[u, v] + \frac{\delta}{M} \|u\|^\frac{1}{2})^\frac{1}{2}.$$ \hspace{1cm} (1.1)

If in such norm the space $D[t]$ is complete (i.e. to be a Banach space) then the sectorial form $t[u, v]$ is called closed form. The following theorem is one of the basic theorems of the theory of m-sectorial forms;

Theorem 2.2. Now we discuss on the relation between the m-sectorial operators and the sesquilinear forms. Let the form $t[u, v]$ be a densely defined closed, sectorial and sesquilinear form in $H$, then there exist an m-sectorial operator $T$ such that

(i) $D(T) = D(t)$ and $t(u, v) = (Tu, v)$. For every $u \in D(T)$ and $v \in D(t)$

(ii) $D(T)$ is a core of $t$;

(iii) if $u \in D(t)$ and $v \in H$ and $t(u, v) = (w, v)$ holds for every $v$ belonging to a core of $t$, then $u \in D(T)$ and $Tu = w$.

The m-sectorial operator $T$ is uniquely determined by the condition (i) and the domain of $T$ is $D(T)$, such that the elements of $u \in D(T)$ satisfy the following inequality

$$\|t[u, v]\| \leq M \|v\|, \quad (v \in D(t)).$$

Notice that by using the above inequality and the density property of $D(t)$ in $H$ there exists a continuous functional $\ell : H \to \mathbb{C}$ such that

$$\ell(v) = \overline{t[u, v]} \quad (v \in D(t)).$$
By using Riesz representation theorem there exists a unique element \( g \in H \) such that for all \( v \in H \) we have \( \langle v, g \rangle \), i.e., \( \langle u, v \rangle = \langle g, v \rangle \), for all \( v \in D(t) \), that implies \( g = T u \).

Theorem 2.3. Let \( h \) be a densely defined, closed and symmetric form that is bounded from below, and let \( H = T \) be the associated self-adjoint operator. The relationship \( h[u, v] = (Hu, v) \) connecting the form \( h \) with the operator \( H \) is unsatisfactory indeed it is not valid for all \( u, v \in D(h) \) because \( D(H) \) is in general a proper subset of \( D(h) \). A more complete representation of \( h \) is furnished by the following theorem.

Theorem 2.4. Let \( h \) be a densely defined, closed and symmetric form, such that \( h \geq 0 \), and let \( H = T \) be the associated self-adjoint operator, then we have \( D(H^+) = D(h) \) and so

\[
1 \cdot h[u, v] = (H^2 u, H^2 v), \quad u, v \in D(h).
\]

A subset \( D' \) of \( D(h) \) is a core of \( h \) if and only if it is a core of \( H^+ \).

Remark. We know that \( H^+ \) is a non-negative self-adjoint operator and \( H \) is an m-accretive operator, such that

\[
(H^+)^* = H.
\]

Theorem 2.5. Let \( T \) be an m-sectorial operator with vertex 0 and semi-angle \( \theta \), then \( H = \text{Re} \, T \) is non-negative and there is a symmetric operator \( B \in \mathcal{B}(H) \), such that \( \|B\| \leq \tan \theta \) and we have

\[
T = G(1 + iB)G^* \quad G = H^+.
\]

Notice that the proof of the above theorems (i.e., Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.4, and Theorem 2.4) are proved in Chapter 6 of Kato (see (Kato, 1966)).

The Basic Properties of M-sectorial Operators:

If \( T \) is an m-sectorial operator with vertex 0, then for every \( \text{Re} \, \lambda > 0 \) the inverse operator \( (T + \lambda \cdot I)^{-1} \) exists such that

\[
\| (T + \lambda I)^{-1} \| \leq \text{Re} \lambda^{-1}
\]

If \( T \) is an m-sectorial operator with vertex 0 and semi-angle \( \phi \in \left( \frac{\pi}{2}, \theta \right] \), then

\[
\| (T - zI)^{-1} \| \leq \frac{1}{|z|} \quad \text{where} \quad |z| > 0.
\]

If in Theorem 2.2, the form \( t[u, v] \) to be symmetric (i.e., \( t[u, v] = t[v, u] \) \( u, v \in D(t) \)), then the operator \( T \) is self-adjoint \( (T = T^*) \) and positive.

For every form \( t[u, v] \) we define the real part of \( t \) by \( t' = \text{Re} \, t \), i.e.,

\[
t'[u, v] = \frac{1}{2} \left[ t[u, v] + t[v, u] \right]. \quad D(t) = D(t').
\]

Suppose \( t[u, v] \) satisfies the conditions of Theorem 2.2, and since the form \( t' = \text{Re} \, t \) also satisfies those conditions since \( t' = \text{Re} \, t \) is symmetric according to Theorem 2.2, then there exists a positive operator \( G = G^2 \geq 0 \) such that \( D(t) \subset D(t') \) and so \( t'[u, v] = (Gu, v) \), for every \( u \in D(G) \) and \( v \in D(t') \), the operator \( G \) exists a positive and unique operator \( Q = Q^2 \geq 0 \) such that \( Q^2 = G \), i.e.,

\[
Q = G^2
\]

therefore

\[
\frac{1}{2} (G^2 u, G^2 v) = t'[u, v], \quad u, v \in D(Q)
\]
Since the form $t[u, v]$ is closed, the operator $G^{1/2}$ is closed and we conclude that

$$D(t) = D(t'), \quad (G'u, G'v) = t'[u, v], \quad u, v \in D(G).$$

For every form $t[u, v]$ we define its adjoint $t'[u, v]$ by

$$t'[u, v] = \overline{t[u, v]}, \quad D(t') = D(t).$$

If the form $t[u, v]$ satisfies the conditions of Theorem 2.2, then the form $t'[u, v]$ satisfies those conditions, therefore there exists an m-sectorial operator $T_t$ such that

$$t'[u, v] = \overline{t[u, v]}, \quad u \in D(T_t'), \quad v \in D(t')$$

It is easy to show that the operator $T_t$ is a self-adjoint operator (i.e. $T = T'$).

**An Important Example of m-sectorial Operator:**

Let $H = L^2(0, 1)$, and define the following norm by

$$|u|_d = \left( \int_0^1 2^\alpha (1-t)^{2\alpha} |\mu'(t)|^2 dt + |\mu(t)|^2 \right)^{1/2}$$

for $0 < \alpha < 1$ by the space $H_\alpha = \tilde{W}_{2\alpha}(0, 1)$ we denote the closure of $C_0^\infty[0,1]$ with respect to the above norm and we have $H = H_1$. Now consider the following bilinear form

$$t[u, v] = \int_0^1 2^\alpha (1-t)^{2\alpha} u(t) u'(t) v(t) \overline{v'(t)} dt,$$

where $D(t) = \tilde{W}_{2\alpha}(0, 1)$, here $\mu(t) \in C^\infty[0,1]$ is a complex satisfying the following conditions

$$\mu(t) \neq 0, \quad |\arg \mu(t)| \leq \theta, \quad (\theta < \frac{\pi}{2})$$

If in the relation (1.1) instead of the form $t$ we set the norm of relation (3.2), then the norm of (1.1) is equivalent to (3.1), since the space $\tilde{W}_{2\alpha}(0,1)$ is a Banach space then $D(t)$ is also Banach, i.e., the form $t[u, v]$ is a closed form. From the relations (3.2), (3.3) we conclude that

$$\{t[u, v] : u \in D(t)\} \subset \{z \in \mathbb{C} : |\arg z| \leq \theta\}.$$

Since the form $t[u, v]$ satisfies the conditions of Theorem 2.2, then there exists an m-sectorial operator $T$ such that

$$D(t) \subset \tilde{W}_{2\alpha}(0,1)$$

and

$$(Tu, v) = \int_0^1 2^\alpha (1-t)^{2\alpha} \mu(t)u'(t) v(t) \overline{v'(t)} dt, \quad u \in D(T), \quad v \in \tilde{W}_{2\alpha}(0, 1).$$

Here $(,)$ is the inner product in $H = L^2(0, 1)$. The m-sectorial operator $T$ satisfying the above conditions is unique. According to Theorem 2.2, the subset $D(t)$ is

$$D(t) = \{u \in \tilde{W}_{2\alpha}(0,1) : |t[u, v]| \leq M_u \|v\|, \quad (v \in \tilde{W}_{2\alpha}(0,1)\}.$$
Let $u \in D(T)$ and $f = Tu$ then

$$ (f, v) = \int_0^1 t^{2\alpha} (1-t)^{2\alpha} \mu(t) u'(t) \overline{v(t)} dt$$

$u \in D(T)$, $v \in C_0^\infty(0,1)$. 

(3.4)

indeed the above equality is an extension of the following function

$$ g = -(t^{2\alpha} (1-t)^{2\alpha} \mu(t) u'(t))' \in D'(0,1). $$

Then $g = f \in L^1(0,1)$, since $u \in W^{1}_{2,\alpha}(0,1)$, then we will have $u' \in L^2_{be}(0,1)$ i.e., $u \in W^{2}_{2,\alpha}(0,1)$.

The space $W^{2}_{2,\alpha}(0,1)$ for $m = 0, 1, 2$ is the space of all functions $u(t)$ ($0 < t < 1$) such that

$$ W^{2}_{2,\alpha}(0,1) = \{ u \in L^1(0,1) : \int_0^1 |x^{(2)}(t)|^2 \, dt < \infty \}, $$

$\varepsilon \in (0, \frac{1}{2}).$

Therefore we proved if $u \in D(T)$ then

$$ u \in W^{1}_{2,\alpha}(0,1) \cap W^{2}_{2,\alpha}(0,1), $$

and

$$ f = -(t^{2\alpha} (1-t)^{2\alpha} \mu(t) u'(t))' \in L^2(0,1). $$

(3.5)

Conversely, if (3.5) and (3.6) are satisfied, then by partial integral, we can also show that (3.4) is satisfied.

Since $C_0^\infty(0,1)$ is dense in $W^{1}_{2,\alpha}(0,1)$ then the equality (3.4) holds (i.e., for every $v \in W^{1}_{2,\alpha}(0,1)$). Now we proved that $D(T)$ is the set of all the functions $u(t)$ ($0 < t < 1$) that satisfies the conditions (3.5), (3.6). If the conditions (3.5), (3.6) are satisfied then $f = Tu$. The adjoint of the form $t$ is defined by

$$ t^*[u, v] = \int_0^1 t^{2\alpha} (1-t)^{2\alpha} \mu(t) u'(t) \overline{v(t)} dt$$

$u \in D(t^*) = W^{1}_{2,\alpha}(0,1).$

According to Theorem 2.2, there exists an $m$-sectorial operator $T$, such that according to Theorem 2.4, it is the adjoint of $T$. Now if we repeat the above operations the following theorem holds

Theorem 3.1. The domain of the operator $T^*$ consists of

$$ u(t) \in W^{1}_{2,\alpha}(0,1) \cap W^{2}_{2,\alpha}(0,1), $$

such that

$$ F = -(t^{2\alpha} (1-t)^{2\alpha} \mu(t) u'(t))' \in L^2(0,1). $$

In Theorem 2.5, we have $F = T' u$. From the other side we have

$$ D(T^*) = \{ u \in W^{1}_{2,\alpha}(0,1) : |t^*[u, v]| \leq M_u \| v \|, \quad v \in C_0^\infty(0,1) \}. $$

For $\varphi \in (0, \pi)$ we have such estimate

$$ \| (T - \lambda I)^{-1} \| \leq M_\varphi |\lambda|^{-1}, \quad (\lambda \in \Phi_\varphi). $$

$$ \| (T' - \lambda I)^{-1} \| \leq M_\varphi |\lambda|^{-1}, \quad (\lambda \in \Phi_\varphi). $$
For every $t[u, v]$ form we define the real part of $t$ equal to $t' = \text{Re} t$ i.e.

$$\ell'[u, v] = t' - \frac{1}{2} \int_0^1 \ell^{2a} (1-t)^{2a} \mu(t) u(t) v(t) dt$$

where

$$L(t') = \tilde{W}_{2a}(0, 1).$$

(3.7)

Here $\mu(t) \in C^{11}[0, 1]$ is a complex function that satisfies the following conditions $\mu(t) = \text{Re} \mu(t)$. According to Theorem 2.2, the form $t_0$ defines the operator $G = G^{0} = 0$. For the form $t$ and the number $\lambda > 0$ the form $t_1$ is defined by

$$t_1[u, v] = t[u, v] + \lambda [u, v], \quad D(t_1) = D(t).$$

Suppose $t$ and $t'$ to be the forms (3.4), (3.7), then the forms $t_1$ and $t_1'$ defines an $m$-sectorial operator $T_1$ and

$$G_1 = G_1^a = G + \lambda I,$$

it is easy to show that

$$T_1 = T + \lambda I, \quad G_1 = G + \lambda I.$$

From Theorem 2.2, we conclude that

$$(T + \lambda I) = (G + \lambda I)^{1/2} (I + i B(\lambda)) (G + \lambda I)^{1/2}, \quad \lambda \geq 0$$

(3.8)

Since $B(\lambda) = B(\lambda')$ is a bounded operator, for every $u \in L^2(0, 1)$ we will have

$$\| (I + i B(\lambda)) u \|^2 = \| u \|^2 + \| B(\lambda) u \|^2 \geq \| u \|^2,$$

i.e.,

$$\| (I + i B(\lambda)) u \|^2 \leq 1.$$  

From here and (3.8) we will have

$$(T + \lambda I)^{-1} = (G + \lambda I)^{-1/2} X(\lambda) (G + \lambda I)^{-1/2}, \quad \| X(\lambda) \| \leq 1, \quad \lambda > 0.$$  

(3.9)

Similarly, we will have

$$(T^* + \lambda I)^{-1} = (G + \lambda I)^{-1/2} X(\lambda)^* (G + \lambda I)^{-1/2}, \quad \| X(\lambda)^* \| \leq 1, \quad \lambda > 0.$$  

These equalities help us to use the properties of the self-adjoint operators. The domain of $G$ denoted by $D(G) = \{ u [t] \in \tilde{W}_{2a}(0, 1) \cap W_{2a,lec}^2 (0, 1) : (\ell^2 (1-t)^{2a} \mu_0 (t) u(t))' \in L^2(0, 1) \}$

If $u \in D(G)$, $G u = -(\ell^2 (1-t)^{2a} \mu_0 (t) u(t))'$, where

$$\mu_0 (t) \in C^{11}[0, 1], \quad c^\prime \leq \mu_0 (t) \leq c, \quad c, c^\prime > 0, \quad G = G^a \geq 0.$$

Many mathematicians work on the operator $G$. For example if $a = 1$ the operator $(G + \lambda I)^{-1}$ on the $L^2(0, 1)$ is compact. Indeed

$$(G + \lambda I)^{-1} \in S(H), \quad (H = L^2(0, 1)).$$

(3.10)
The space $S_1(H)$ induces from the operators $Q$ is such that
$$|\mathcal{E}| = \sum_{i=1}^{n} \lambda_i(\mathcal{E} \cdot \mathcal{Q}^{-1}) < \infty,$$

here $\lambda_i(.)$, $i = 1, 2, \ldots$ are the eigenvalues of the operator $Q'Q$. If $Q_1$, $Q_2 \in S_1(H)$ are bounded operators, then we will have
$$|e_1 e_2| \leq |e_1| |e_2|, \quad |e_2 e_1| \leq |e_1| |e_2|.$$

If $Q \in S_1(H)$, then
$$\sum_{i=1}^{n} |\lambda_i(\mathcal{E})| \leq |\mathcal{E}|.$$

And the trace of the function $Q$ is denoted by
$$tr Q = \sum_{i=1}^{n} \lambda_i(\mathcal{E}).$$

(3.11)

From (2.10) and (2.11), we conclude that the operator $(T + \lambda I)^{-1}$ is compact, then the operator $T$ has a countable spectrum and the eigenvalues of the operator $(T + \lambda I)^{-1}$ denoted by $$(\lambda_1(T) + \lambda)^{-1}, (\lambda_2(T) + \lambda I)^{-1},$$

from (3.8) and (3.11) we conclude that
$$\sum_{i=1}^{n} |\lambda_i(T) + \lambda| \leq \|(T + \lambda I)^{-1}\| \leq \|(G + \lambda I)^{-1}\|.$$  

Here $\|\cdot\|$ is Hilbert Schmidt norm and we use the inequality
$$|e_1 e_2 e_3| \leq |e_1| |e_2| |e_3|.$$

Since for each $u \in D(T)$, $|\arg(Tu,u)| \leq \theta$, then $|\arg \lambda_i(T)| \leq \theta$, $i= 1,2, \ldots$ i.e.,

$$|\lambda_i(T)| \leq \lambda^0_1 \|A(T) + \lambda\|^{-1}.$$

The following functions are defined by
$$N(t) = \text{card} \{ j : |\lambda_j(T)| \leq t \}, \quad n(t) = \text{card} \{ j : |\lambda_j(\mathcal{Q})| \leq t \},$$

it is easy to show that
$$n(t) \leq M_{1} (1 + t)^2, \quad t \leq 0$$

from the above relations we will have
$$N(t) - \int_0^t dN(s) \leq 2 \int_0^t (\mathcal{Q} + \lambda)^{-1} dN(s)$$

$$\leq 2 \int_0^t (\mathcal{Q} + \lambda)^{-1} dN(s) - 2 \sum_{i=1}^{n} |\lambda_i(T)| \leq 2 M_{1,2} (\mathcal{Q} + \lambda t)^{-1}.$$

On the other hand we have

4565
Now the following theorem is proved

Theorem 3.2. The eigenvalues of the operator $T$ are in the sector $S = \{ z \in \mathbb{C} : | \arg z | \leq \theta \}$, these eigenvalues are in $S = \{ z \in S : |z| \leq t \}$ and are less than $M'(1+t)^{\frac{1}{2}}$ where $M > 0$ is independent of $t$.

Let $N(\xi) = \text{card} \{ j : (\lambda_j(T)) \in S_+ \} \leq M'(1+t)^{\frac{1}{2}}$.

In the end we will speak about the concept of the Friedrichs extension. Consider the operator $T_0$ (in Hilbert space) that satisfies the following conditions:

$$\| \langle T_0 u, v \rangle \| \leq \theta, \quad u \in D(T_0), \quad \theta \in (0, \frac{\pi}{2}).$$

If $D(T_0)$ is dense in $H$ then we denote the closure of $D(T_0)$ in the following norm

$$\| u \|_+ = (\Re \langle T_0 u, v \rangle + \| u \|^2)^{\frac{1}{2}}.$$  \hspace{1cm} (3.1)

Let $H_+ = D(\cdot)$ then we defined the bilinear form $\langle u, v \rangle$ for $u \in D(T)$, $v \in H$, by;

$$\langle u, v \rangle = \langle T_0 u, v \rangle.$$

If $u \in H \setminus D(T_0)$, then there exist the sequence $\{ u_n \} \subset D(T_0)$ such that in $H$, $u_n \to u$. In this case we set

$$\langle u, v \rangle = \lim_{n \to \infty} \langle T_0 u_n, v \rangle.$$

It is easy to see that the form $\langle u, v \rangle$ is a closed sectorial bilinear form and also $D(\cdot)$ is dense in $H$.

According to Theorem 2.2 there exists an m-sectorial operator $T$ such that

(i) $DT = D(\cdot)$ and

$$\langle u, v \rangle = \langle Tu, v \rangle.$$

(ii) $D(T) = \{ u \in H_+ : \| u \|_+ \leq M_u \| v \|, \quad (v \in H_+) \}.$$

The operator $T$ is unique and is called the Friedrichs extension of $T_0$. It is easy to see that the differential operator $T$ that we got above is the Friedrichs extension of the following operator

$$T_0 u = -u^{(2)} (1-t)^{2t} \mu(t) u'(t),$$

with domain $D(T_0) = C^\infty_0(0,1)$.

REFERENCES


Kh, K., Boimatov and A.G. Kostychenko, 1990. The spectral asymptotics of non-selfadjoint elliptic


