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Three New Iterative Methods for Solving Nonlinear Equations

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Abstract: In this paper, we present a family of new iterative methods for solving nonlinear equations based on Newton's method. The order of convergence and corresponding error equations of the obtained iteration formulae are derived analytically and with the help of Maple. Some numerical examples are given to illustrate the efficiency of the presented methods, so one would be able to compare the results of the same problems obtained by applying different methods, and the advantage of the new methods can be recognized.

Key words: Newton's method, order of convergence, iterative methods, Maple

INTRODUCTION

We consider the problem of finding a real simple zero a of a nonlinear equation $f(x)=0$. This problem has good applications in science and engineering. Much attention has given to develop several iterative methods for solving nonlinear equations for solving nonlinear equations, see {Abbasbandy (2003), Chun (2005), Homeier (2005), Noor (2007), Stoer and Bulirsch (1993), Weerakoon and Fernando (2000)} and the references therein.

One of the well-known methods for solving a single nonlinear equation $f(x)=0$ is the classical Newton's method (Stoer and Bulirsch (1993)) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, ... \tag{1}$$

This method converges quadratically in some neighborhood of a simple root a of *f*.

The solution of the equation $M(x)=0$ is denoted x_{n+1} where

$$M(x) = f(x_n) + f'(x_n)(x - x_n). \tag{2}$$

If the integral, arising from Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt, \tag{3}$$

approximated by the rectangular rule (Stoer and Bulirsch (1993)), i.e.

$$\int_{x_n}^{x} f'(t)\,dt \approx f'(x_n)(x - x_n),$$

we obtain the linear equation (2).

Weerakoon and Fernando (2000) rederive the Newton's method by approximating the integral by the rectangular rule. When they used the trapezoidal approximation (Stoer and Bulirsch (1993))

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$$\int_{x_n}^{x} f'(t)dt \approx (x - x_n)\left(\frac{f(x_n) + f'(x_n)}{2}\right),$$

in combination with the approximation $f'(x) \approx f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$, they arrived at the modified Newton-type iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}, n = 0, 1, ... \tag{4}$$

They proved that this method converges of order.

Here we continue this approach to receive some new iterative methods on the same problem.

New Iterative Methods:

We approximate the integral in (3) by modified trapezoid method (Stoer and Bulirsch (1993)) which is:

$$\int_{x_n}^{x} f'(t)dt \approx \frac{(x - x_n)}{2}[f'(x_n) + f'(x)] + \frac{(x - x_n)^2}{12}[f''(x_n) - f''(x)]. \tag{5}$$

We obtain the equation

$$M_n(x) = f(x_n) + \frac{(x - x_n)}{2}[f'(x_n) + f'(x)] + \frac{(x - x_n)^2}{12}[f''(x_n) - f''(x)]. \tag{6}$$

We find the solution x_{n+1} of the equation $M_n(x_{n+1}) = 0$

$$f(x_n) + \frac{(x_{n+1} - x_n)}{2}[f'(x_n) + f'(x_{n+1})] + \frac{(x_{n+1} - x_n)^2}{12}[f''(x_n) - f''(x_{n+1})] = 0,$$

and obtain

$$x_{n+1} = x_n - \frac{12f(x_n) + (x_{n+1} - x_n)^2[f''(x_n) - f''(x_{n+1})]}{6[f'(x_n) + f'(x_{n+1})]}, \tag{7}$$

If we approximate $(x_{n+1} - x_n)$ in the right hand side of (7) by a Newton's method (1), yields

$$x_{n+1} = x_n - \frac{12f(x_n)f'^2(x_n) + f^2(x_n)[f''(x_n) - f''(x_{n+1})]}{6f'^2(x_n)[f'(x_n) + f'(x_{n+1})]}. \tag{8}$$

Formula (8) is an implicit formula. We replace x_{n+1} on the right hand side of (8) by the Newton's iterative method (1) to obtain a new iterative method with third order convergence

$$x_{n+1} = x_n - \frac{12f(x_n)f'^2(x_n) + f^2(x_n)\left[f''(x_n) - f''(x_n - \frac{f(x_n)}{f'(x_n)})\right]}{6f'^2(x_n)\left[f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]}, \quad n = 0, 1, ...$$

Also, if we approximate $(x_{n+1} - x_n)$ in the right hand side of (7) by a Halley method (Noor (2007)):

$$x_{n+1} - x_n = -\frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \tag{10}$$

We obtain another new formula with fourth order convergence

$$x_{n+1} = x_n - \frac{12f(x_n)f'^2(x_n) + f^2(x_n)[f''(x_n) - f''(y_n)]}{6f'^2(x_n)[f'(x_n) + f'(y_n)]}, \quad n = 0, 1, \dots \tag{11a}$$

where

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}. \tag{11b}$$

Now, to derive another iterative method, we combining (9) with (1) as follows:

$$z_n = x_n - \frac{12f(x_n)f'^2(x_n) + f^2(x_n)\left[f''(x_n) - f''(x_n - \frac{f(x_n)}{f'(x_n)})\right]}{6f'^2(x_n)\left[f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{12}$$

This method converges of order six.

Convergence Analysis:

In this section we consider the convergence analysis of the new introduces methods (Equations (9), (11) and (12)).

Theorem 1:

Assume that the function $f : I \rightarrow R$ for an open interval I has a simple root $\alpha \varepsilon I$. If $f(x)$ is sufficiently smooth on the neighborhood of the root α , then the iterative method defined by (9) converges of order three.

Proof:

Let $e_n = x_n - \alpha$, using Taylor's expansion about α, we can write

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + \cdots], \tag{13}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + \cdots], \tag{14}$$

$$f''(x_n) = f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + \cdots], \tag{15}$$

where

$$c_j = \frac{f^j(\alpha)}{j!\,f'(\alpha)}, j = 2, 3, \dots .$$

Dividing (13) by (14) gives

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (-3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 + O(e_n^5). \quad (16)$$

Using equations (1) and (16), we get

$$y_n = c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \quad (17)$$

From (17), by using Taylor series about α, we have

$$f(y_n) = c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \quad (18)$$

$$f'(y_n) = 1 + 2c_2^2 e_n^2 - 4(-c_3 + c_2^2)c_2 e_n^3 + O(e_n^4), \quad (19)$$

$$f''(y_n) = 2c_2 + 6c_2c_3 e_n^2 - 12(-c_3 + c_2^2)c_3 e_n^3 + O(e_n^4). \quad (20)$$

From (13), (14), (15) and (20), we obtain

$$k_1 = 12e_n + 60c_2 e_n^2 + 12(90c_3 + 96c_2^2)e_n^3 + O(e_n^4), \quad (21)$$

where

$$k_1 = 12f(x_n)f'^2(x_n) + f^2(x_n)[f''(x_n) - f''(y_n)].$$

From (14) and (19), we obtain

$$k_2 = 12 + 60c_2 e_n + (90c_3 + 108c_2^2)e_n^2 + \cdots + O(e_n^4), \quad (22)$$

where

$$k_2 = 6f'^2(x_n)[f'(x_n) + f'(y_n)]$$

Using (21) and (22) we obtain

$$k_3 = e_n - c_2^2 e_n^3 + \left(-\frac{7}{2}c_2c_3 3 + c_2^3\right)e_n^3 + O(e_n^4), \quad (23)$$

where

$$k_3 = \frac{k_1}{k_2}.$$

Finally from (9) and (23) we obtain

$$e_{n+1} + \alpha = e_n + \alpha - \left[e_n - c_2^2 e_n^3 + \left(-\frac{7}{2}c_2c_3 3 + c_2^3\right)e_n^4 + O(e_n^4)\right].$$

Thus,

$$e_{n+1} = c_2^2 e_n^3 + \left(\frac{7}{2}c_2c_3 - 3c_2^3\right)e_n^4 + O(e_n^4).$$

This means that the iterative method defined by (9) has third order convergences. ■

Also we find the order convergence of (9) by using Maple as follows:

```
> y:=x->x-f(x)/(D(f)(x));
```

$$y := x \to x - \frac{f(x)}{f'(x)}$$

```
> z:=x->x-(12*f(x)*D(f)(x)^2+f(x)^2*(D(D(f))(x)-
(D (D (f))@y)(x)))/(6*D(f)(x)^2*(D(f)(x)+(D(f)@y)(x)));
```

$$z := x \to x - \frac{12f(x)D(f)(x)^2 + f(x)^2(D(D(f))(x) - D(D(f))@y(x))}{6D(f)(x)^2(D(f)(x) + D(f)@y(x))}$$

```
> algsubs (f (α) =0, z (α));
```

α

```
> algsubs (f (α) =0, D (z) (α));
```

0

```
> algsubs (f (α) =0,D(D (z)) (α));
```

0

```
> algsubs (f (α) =0, (D@@3) (z) (α)));
```

$$\frac{3D^{(2)}(f)(\alpha)^2}{2D(f)(\alpha)^2}$$

Thus, we obtain

$$z(\alpha) = \alpha, \quad z'(\alpha) = z^{(2)}(\alpha) = 0,\ z^{(3)}(\alpha) = \frac{3D^{(2)}(f)(\alpha)^2}{2D(f)(\alpha)^2}$$

So, it is of order three.

Theorem 2.

Assume that the function $f: I \to R$ for an open interval I has a simple root $\alpha \in I$. If $f(x)$ is sufficiently smooth on the neighborhood of the root α, then the iterative method defined by (11) converges of order four.

Proof:

From (11b) we have

$$y_n = (-c_3 + c_2^2)e_n^3 + (-3c_4 + 6c_2c_3 - 3c_2^3)e_n^4 + O(e_n^5). \quad (24)$$

By using Taylor series about α,we have

$$f(y_n) = (-c_3 + c_2^2)e_n^3 + (-3c_4 + 6c_2c_3 - 3c_2^3)e_n^4 + O(e_n^5). \quad (25)$$

$$f'(y_n) = 1 + 2(-c_3 + c_2^2)c_2e_n^3 - \quad 6c_2(c_4 - 2c_2c_3 + c_2^3)e_n^4 + O(e_n^5), \quad (26)$$

and

$$f''(y_n) = 2c_2 + 6(-c_3 + c_2^2)c_3e_n^3 - \quad 18c_3(c_4 - 2c_2c_3 + c_2^3)e_n^4 + O(e_n^5). \quad (27)$$

From (13), (14), (15), (26) and (27), we obtain

$$l_1 = 12e_n + 60c_2e_n^2 + (90c_3 + 96c_2^2)e_n^3 + \cdots + O(e_n^5), \tag{28}$$

where

$$l_1 = 12f(x_n)f'^2(x_n) + f^2(x_n)[f''(x_n) - f''(y_n)].$$

From (14) and (26), we obtain

$$l_2 = 12 + 60c_2e_n + (90c_3 + 96c_2^2)e_n^2 + \cdots + O(e_n^5), \tag{29}$$

where

$$l_2 = 6f'^2(x_n)[f'(x_n) + f'(y_n)].$$

Using (28) and (29) we get

$$l_3 = e_n - c_2^3e_n^4 + O(e_n^5), \tag{30}$$

where

$$l_3 = \frac{l_1}{l_2}.$$

Finally from (11) and (30) we obtain

$$e_{n+1} + \alpha = e_n + \alpha - [e_n - c_2^3e_n^4 + O(e_n^5)].$$

Thus,

$$e_{n+1} = c_2^3e_n^4 + O(e_n^5). \tag{31}$$

This means that the iterative method defined by (11) has fourth order convergences. ■

Theorem 3.

Assume that the function $f: I \to R$ for an open interval I has a simple root $\alpha \in I$. If $f(x)$ is sufficiently smooth on the neighborhood of the root α, then the iterative method defined by (12) converges of order six.

Proof:

From (23) we have

$$k_3 = e_n - c_2^2e_n^3 + \left(-\frac{7}{2}c_2c_3 + 3c_2^3\right)e_n^4 + O(e_n^5). \tag{32}$$

Then

$$z_n = c_2^2e_n^3 + \left(\frac{7}{2}c_2c_3 - 3c_2^3\right)e_n^4 + O(e_n^5). \tag{33}$$

Now expanding *f(zn)* about α, and using (33) we have

$$f(z_n) = c_2^2 e_n^3 + \left(\frac{7}{2} c_2 c_3 - 3c_2^3\right) e_n^4 + O(e_n^5), \tag{34}$$

$$f'(z_n) = 1 + 2c_2^3 e_n^3 - c_2^2(-7c_3 + 6c_2^2) e_n^4 + O(e_n^5). \tag{35}$$

Finally form (33), (34) and (35) we obtain

$$e_{n+1} = c_2^5 e_n^6 + O(e_n^7). \tag{36}$$

This means that the iterative method defined by (12) has sixth order convergences.

Also we find the order of convergence of (12) by using Maple as follows:

```
> y:=x->x-f(x)/(D(f)(x));
```

$$y := x \to x - \frac{f(x)}{f'(x)}$$

```
> z:=x->x-(12*f(x)*D(f)(x)^2+f(x)^2*(D(D(f))(x)-
(D (D (f))@y)(x)))/(6*D(f)(x)^2*(D(f)(x)+(D(f)@y)(x)));
```

$$z := x \to x - \frac{12f(x)D(f)(x)^2 + f(x)^2(D(D(f))(x) - D(D(f))@y(x))}{6D(f)(x)^2(D(f)(x) + D(f)@y(x))}$$

```
> w:=x->w(x)-(f@w)(x)/(D(f)@w)(x);
```

$$w := x \to z(x) - \frac{f@z(x)}{D(f)@z(x)}$$

```
> algsubs (f (α) =0, w (α));
                                        α
> algsubs (f (α) =0, D (w) (α));
                                        0
> algsubs (f (α) =0,D(D (w)) (α));
                                        0
> algsubs (f (α) =0, (D@@3) (w) (α)));
                                        0
> algsubs (f(a)=0,(D@@4)(z)(a));
                                        0
> algsubs (f(a)=0,(D@@5)(z)(a));
                                        0
> algsubs (f(a)=0,(D@@6)(z)(a));
```

$$\frac{42D^{(2)}(f)(\alpha)^5}{2D(f)(\alpha)^5}$$

Thus, we obtain

$$z(\alpha) = \alpha, \quad z'(\alpha) = z^{(2)}(\alpha) = z^{(3)}(\alpha) = z^{(4)}(\alpha) = z^{(5)}(\alpha) = 0,$$

and

$$z^{(6)}(\alpha) = \frac{3D^{(2)}(f)(\alpha)^2}{2D(f)(\alpha)^2}$$

So, it is of order six.

Numerical Examples:

We present some numerical examples to illustrate the efficiency of the new iterative methods (Table 1). We compare the Newton's method (1) (NM), the method of Abbasbandy (AM) (Abbasbandy (2003)), the method of Homeier (HM) (Homeier (2005)), the method of Chun (CM) (Chun (2005)), and our new methods (9) (N1), (11) (N2) and (12) (N3) .

We do all computations by using Maple 9.5. We accept an approximate solutions rather than the exact root, depending on the precision e. We use the following stopping criteria are used for computer programs:

$$|f(x_n)| < 10^{-15}$$

$$|x_{n+1} - x_n| < 10^{-15}$$

We used the following test functions and displays the computed approximate zero x_* using 27th decimal place.

$$f_1(x) = cosx - x, \qquad x_* = 0.739085133215606416553120876$$

$$f_2(x) = x^3 + 4x^2 - 10, \quad x_* = 1.365230013414096845760806829$$

$$f_3(x) = sinx - \frac{x}{2}, \qquad x_* = 1.895494267033980947144035738$$

$$f_4(x) = (x+2)e^x - 1, x_* = -0.442854401002388583141328000$$

$$f_5(x) = (x-1)^3 - 1, \quad x_* = 2$$

$$f_6(x) = x^2 - e^x - 3x + 2, x_* = 0.257530285439860760455367304$$

As convergence criterion, it was required that the distance of two consecutive approximations for the zero was less than 10^{-15} The results presented in Table1 are the number of iterations (IT) to approximate the zero by different methods.

Table 1: Shows a comparison between the iterative methods NM, AM, HM, CM and our new methods and our new methods N1, N2 and N3.

Functions	x_0	IT						
		NM	AM	HM	CM	N1	N2	N3
f_1	1.7	5	4	4	4	3	3	2
f_2	3	6	4	4	4	4	3	3
f_3	2.3	5	4	3	5	3	3	2
f_4	2	8	6	5	6	5	4	4
f_5	3.5	6	4	4	5	5	4	3
f_6	2	6	5	5	4	3	3	2

Conclusion:

In this paper, we suggest and analyze three iterative methods for solving nonlinear equations. We observed from numerical examples that the proposed methods have at least equal performance as compared with the other methods in Table 1.

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