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Exact and Numerical Solution of Pure Torsional Shaft

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Abstract: Based on the exact and numerical analysis on complex structure, we analyzed characteristic of maximum shear stress on boundary cross sectional that is closest from centrepoint of torsion (Gravity Centre). This paper deals with the comparison of exact and numeric solution on pure torsion shaft which holds torsion 2,5 Nm, dimension is major axis, *a* and minor axis, *b* are 1.2375 x 10^{-2} m and 1.05 x 10^{-2} m, and prismatic length, *l* is 9.845 x 10^{-2} m, respectively. The mechanical properties of the torsional shaft such as shear modulus, *G,* Young modulus, *E,* yield point, $_{\sigma Yield,}$ are considered as 8.02 x 10^{11} Pa, 2.07 x 10 11 Pa, 4.14 x 108 Pa, respectively. The Poisson and Hardening ratio are as 0.29 and 800, respectively. It is found that the exact and finite element analysis have the same characteristic of maximum shear stress on boundary cross sectional that are closest from centre point of torsion i.e. gravity of the centre. This comparative study explored the exact simulation and numerical simulation by FEM has the divergent deviation to maximum shear stress.

Key words: FEM, pure torsional shaft, exact solution

INTRODUCTION

Analysis of complex structure is importance both exact and numerical solution. On the exact solution needs a long complicated mathematical differential solution. Instead of this, another method called finite element method is introduced i.e. a numerical solution undergone by discretion structure of infinite in to finite element that continuously build mesh.

Application of pure torsion on prismatic can be done to a certain part of cross sectional of shaft loaded by torque subject to couple transmission, therefore in its manufacturing and fabrication process, stress analysis is significant factor to take into consideration since one of shaft failures may be caused by excessive stress distribution on some area.

The objective of this paper is the comparison of exact solution and finite element for pure torsional shaft. The underlying of the stress analysis of elliptical prismatic with boundary cross sectional area are based on exact solution for three dimensional solid for construction of finite element shaft model.

Basic Equation of Torsion:

Elasticity of Torsion:

Elasticity of a single crystal is unequal in the different direction and random. To achieve a usable homogenous assumption at high accuracy, the elasticity of the geometric element to be average property of crystal. When the geometric direction of the single crystal is different, the material can be considered as an isotropic material.

The Cartesian coordinate of the displacement of cross sectional prismatic area is loaded by torsional approach as in by Saint-Venant method based on equation (1) shown in Figure 1.

$$\begin{aligned} u &= -\beta\, y z \\ v &= \beta\, x z \\ w &= \beta \phi(x, y) \end{aligned} \quad (1)$$

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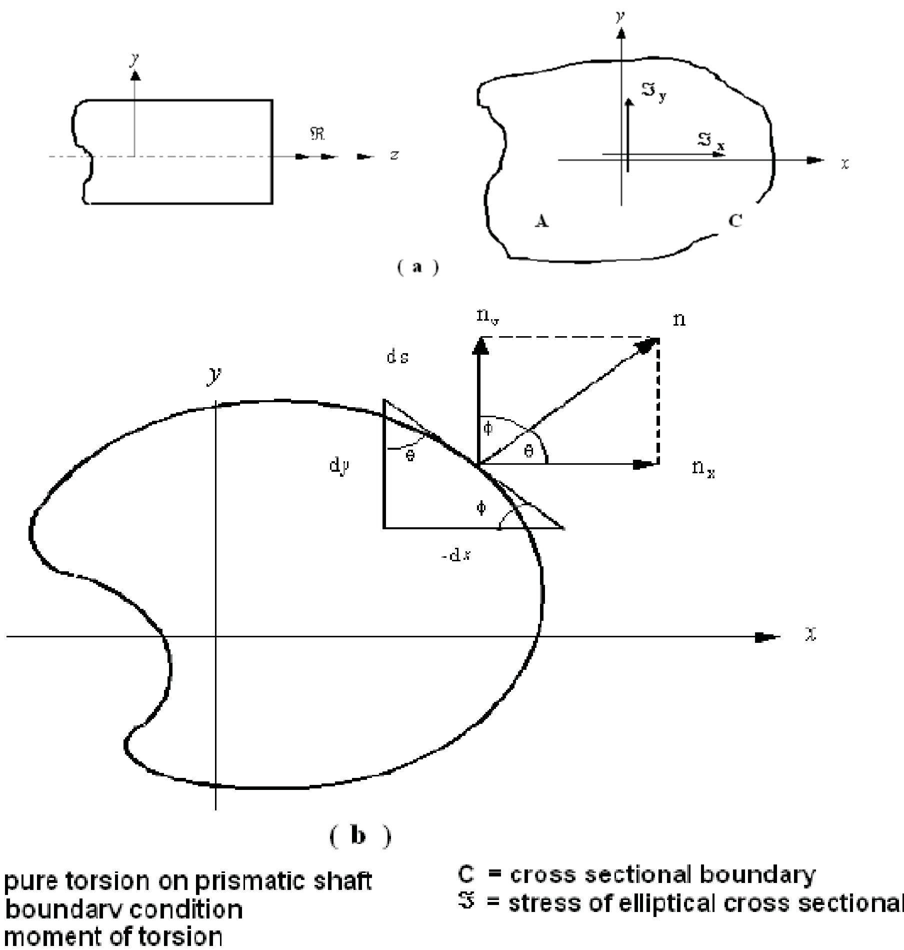
Where u, x direction displacement ; v, y direction displacement ; w, z direction displacement ; β, twist angle and φ (x, y), warping function.



$$\theta + \Phi = \frac{\pi}{2} \quad , \quad n_x = \cos\theta = \frac{dy}{dx} \quad , \quad n_y = \cos\varphi = \sin\theta = \frac{ds}{dx}$$

Fig. 1: Twisting of Prismatic

Eulerian and Lagrangian illustration of linearization characteristic of strained component to displacement value, ε and strained component to shear strain, γ are as follows.

$$2L_{i,j} \cong \gamma_{i,j} \cong 2\varepsilon_{i,j}$$

where

$$\gamma_{i,j} = \frac{\partial u_i(a)}{\partial a_i} + \frac{\partial u_j(a}{\partial a_i} \tag{2}$$

The linearization of Lagrangian rotation tensor on rotation component is as follows,

$$\omega_{i,j} = \frac{1}{2}\left(\frac{\partial u_i(a)}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)$$

$$\omega_{i,j} = -\omega_j$$

(3)

The stress and elasticity constant of the material on pure torque is determined by strain component in which the influential shear stress existed perpendicular to prismatic axis. This is possible in consideration of both strain to displacement value and occurred shear strain can be neglected.

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0$$

$$\tau_{xz} = G\beta\left(\frac{\partial \phi}{\partial x} - y\right) \quad (4)$$

Without considering weight force of prismatic and eliminating stress function on element equilibrium, the boundary condition on cross section area and cross sectional boundary.

$$\Delta^2 \phi = 0$$
$$n \,.\, \Delta \phi = \frac{d\phi}{dn} \quad (5)$$

Where Δ is the differential function, Φ is the warping function and $d\Phi/dn$ is the gradient of w on prismatic cross section.
Relationship between twisting moment, $\Re$ and twist angle, β are as follows.

$$\Re = \int_A (x\tau_{yz} - y\tau_{xz})dA$$
$$= \beta D \quad (6)$$

Where D is the torsion stiffness of prismatic cross sectional.

Exact Analysis:

Based on the definition of stress function with complex variable of the torsional shaft as follows.

$$F(z) = \Phi + i\Psi \quad (7)$$

Where ψ is the complex function
Hence shear strain becomes

$$\tau_{xz} = G\beta\left(\frac{\partial \psi}{\partial y} - y\right)$$
$$\tau_{yz} = G\beta\left(x - \frac{\partial \psi}{\partial x}\right) \quad (8)$$

Stress function according to Ludwig Prandth is,

$$\Phi = \psi - \frac{1}{2}\left(x^2 + y^2\right) \quad (9)$$

Hence boundary condition of cross sectional area (A) and boundary of cross sectional (C)

$$\nabla^2\Phi = -2$$
$$\Phi = \mathrm{K}$$

(10)

The stiffness torsion of elliptical cross sectional area is as follows.

$$D = G\int_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] dA \tag{11}$$

While the Figure 2 shows the stress analysis of elliptical prismatic with boundary cross sectional area based on equation (12) as follows.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12}$$

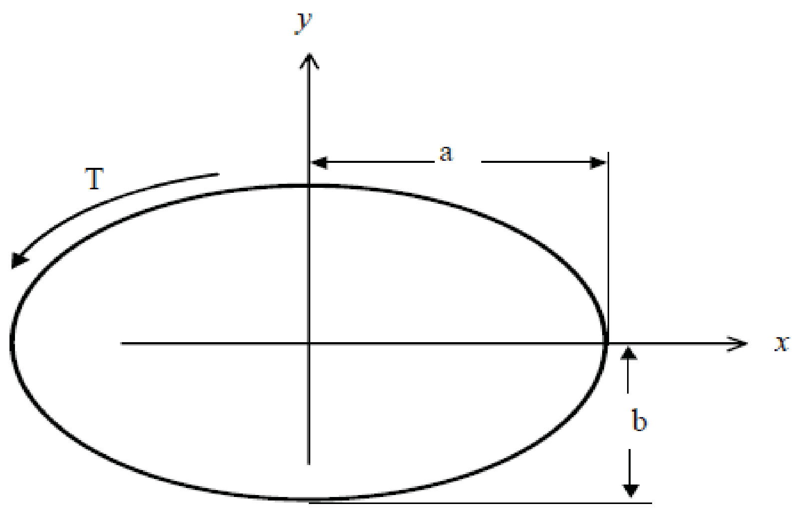


Fig. 2: Cross Sectional of Elliptical Prismatic

The stress function on elliptical cross sectional boundary according to Ludwig Pradath is,

$$\Phi = K\left[\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right)\right]$$

$$\Phi_A = -\frac{b^2x^2 + a^2y^2}{a^2 + b^2} \tag{13}$$

Elliptical cross sectional stiffness torsion is,

$$D = \frac{G\,\pi\,a^3b^3}{a^2 + b^2} \tag{14}$$

It is found that coordinated tangential stress on cross sectional boundary, c is the result of the Pythagoras sum of both shear stresses in square root and its resultant angle as equal as the result of both shear stresses.

$$\tau = \frac{2G\beta ab}{a^2 + b^2}\left(\frac{b^2x^2}{a^2} + \frac{a^2y^2}{b^2}\right)^{\frac{1}{2}}$$

(15)

FEM Analysis:

Finite element technique involves element-modeling discretion, which is defined through a displacement function of each node.

$$\{F\} = [k]\{D\} \tag{16}$$

Modeling is used rectangular trilinear element which has 27 nodes. As the result, the pure torsion is occurred to the prismatic. Accordingly the relation between the outcoming strain matrix, $\varphi^{(e)}$, shape function matrix, *N,* and stiffness matrix, *K* are as follows.

$$\{\phi^{(e)}\} = [\ N_1\ N_2\ \dots\ N_n\ ]\{\Phi^{(e)}\ \}$$
$$[K^{(e)}] = \sum_{i=1}^{N} [k] \tag{17}$$

Next, we can determine the possible outcoming stress by,

$$\{\tau^{(e)}\} \ = [\ B\ ]\{\Phi^{(e)}\} \tag{18}$$

Where $\varphi^{(e)}$ is the displacement matrix.

RESULT AND DISCUSSION

Finite element analysis is supported by FAST Software and structure analysis using FEM 3dat.C. The boundary conditions of prismatic are clamped and torque. This loading characteristic of torque is transformed into concentrated forces, where each has equal torsion on each node. Therefore desirable result of the resultant torsion is shown in Figure 3.

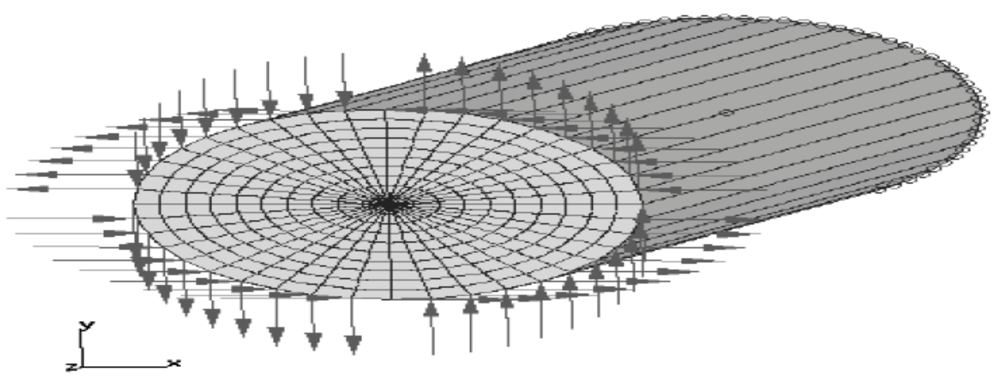


Fig. 3: Boundary Condition of Elliptical Prismatic

Table 1 shows the comparison of exact and numeric solution on elliptical prismatic, which holds torsion 2.5 Nm and dimension of the major axis (*a*) and minor axis (*b*) are as 1.2375 x 10^{-2} m and 1.05 x 10^{-2} m, respectively. The prismatic length, *l* is 9.845 x 10^{-2} m. The mechanical properties of shear modulus, *G,* Young modulus, *E,* yield point, $\sigma_{Yield,}$ are 8.02 x 10^{11} Pa, 2.07 x 10 11 Pa, 4.14 x 1 0^8Pa, respectively. The Poisson ratio,$\tilde{o}$, and hardening ratio are 0.29 and 800, respectively.

Figure 4 and 5 Show the shear stress direction in the *z*-direction. From this analysis we take shear stress on each node for meridional angle (radian). And after that we are comparing the value of each shear stress on the node with exact solution and we shown in Figur 6.

Table 1: Comparison of Exact and Finite Element result

MERIDIONAL ANGLE(°)	SHEAR STRESS (MPa)			
	EXACT	FEM		
		τ_{yz}	τ_{xy}	τ
0,000000	*0,989 787*	-0,020708	0,1554892	*0,156862*
7,469698	*0,993678*	-0,202317	0,1575184	*0,256407*
15,413007	*1,008002*	-0,378269	0,1450308	*0,405119*
23,380027	*1,028703*	-0,544416	0,1193671	*0,557348*
31,652126	*1,054208*	-0,696980	0,0830607	*0,701912*
40,314084	*1,081777*	-0,832578	0,0393264	*0,833506*
49,427157	*1,1 08660*	-0,948171	-0,0082670	*0,948207*
59,014652	*1,132364*	-1,041099	-0,0559881	*1,042600*
69,046075	*1,150829*	-1,109095	-0,0999039	*1,113590*
79,426391	*1,1 62527*	-1,150319	-0,1357099	*1,158300*
90,000000	*1,1 66535*	-1,163527	-0,1589426	*1,1 74330*

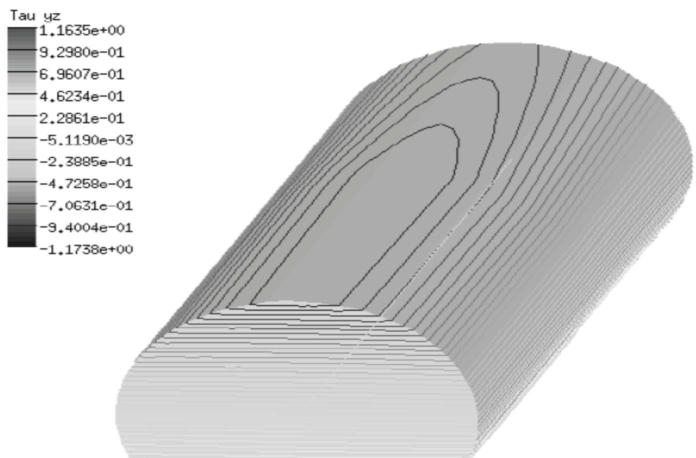


Fig. 4: Shear Stress of yz Direction (τ_{yz})

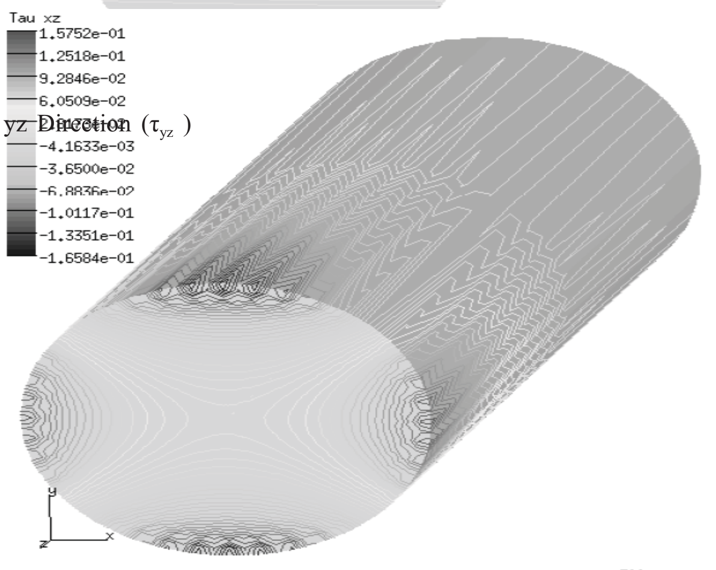


Fig. 5: Shear Stress of xz Direction (τ_{xz})

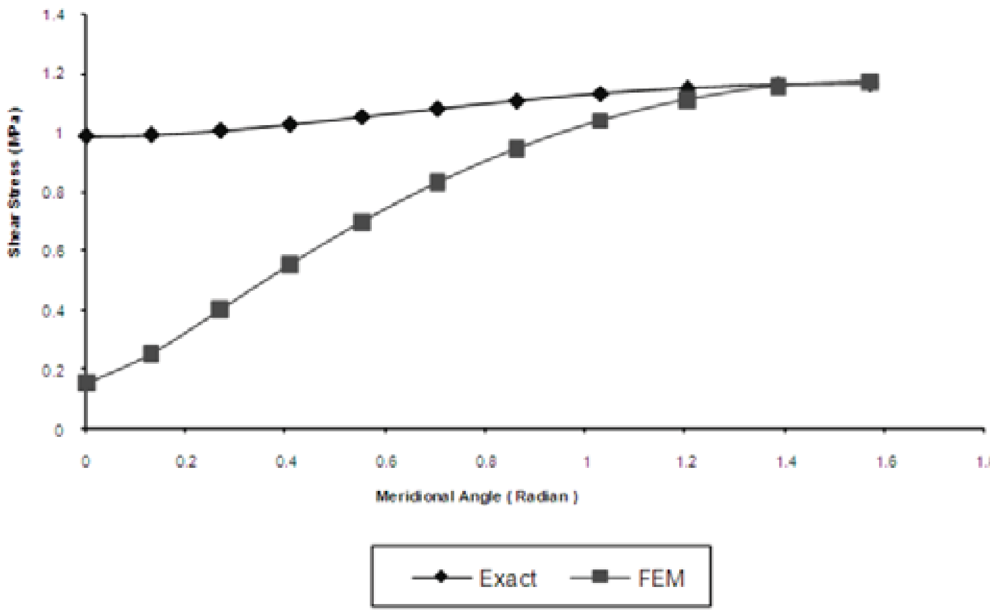


Fig. 6: Relationship Between Exact and Numeric

Figure 4 and 5 Show the shear stress direction in the *z*-direction. From this analysis we take shear stress on each node for meridional angle (radian). And after that we are comparing the value of each shear stress on the node with exact solution and we shown in Figur 6.

Appendix III shows the example of exact solution and appendix IV numeric of Finite Element analysis supported by FAST software.

Summary and Concluding Remarks:

Exact and Finite Element analysis have the same characteristic of maximum shear stress on boundary cross sectional that is closest from centrepoint of torsion (Gravity Centre)

Comperative exact result to FEM has divergent deviation to maximum shear stress.

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Appendix I

$$z = x + i y$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$K = -\frac{a^2 b^2}{\left(a^2 + b^2\right)}$$

Node displacement

$$\upsilon = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 xz + a_8 xyz$$

$$\nu = a_9 + a_{10} x + a_{11} y + a_{12} z + a_{13} xy + a_{14} yz + a_{15} xz + a_{16} xyz$$

$$w = a_{17} + a_{18} x + a_{19} y + a_{20} z + a_{21} xy + a_{22} yz + a_{23} xz + a_{24} xyz$$

$$\{\Phi^{(e)}\} = [K^{-1}]\{f^{(e)}\} \quad \text{(ref. 4, page. 104)}$$

$$\{f^{(e)}\} = \int_A 2G\beta(N)^T dA \quad \text{(ref. 4, page. 91)}$$

$$[k^{(e)}] = \int [b]^T [D] dA \quad \text{(ref. 4, page. 91)}$$

$$[B] = \begin{bmatrix} \dfrac{\partial (N)_{i,j,k}}{\partial x} \\ \dfrac{\partial (N)_{i,j,k}}{\partial y} \end{bmatrix} \quad \text{(ref. 4, page. 91)}$$

$$[D] = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix} \quad \text{(ref. 4, page. 91)}$$

Appendix II (List of Symbol)

u, *x direction displacement*
v, y direction displacement
w, *z direction displacement*
β, twist angle
φ (x,y), warping function
$L_{i,j}$, Lagrangian tensor

τ , shear stress
G, shear modulus
∇, differential function

$\frac{d\phi}{dn}$, gradient of w on prismatic cross sectional

K, stress constant
$\Re$, twisting moment
D, torsion stiffness of prismatic cross sectional
a, major axes
b, minor axes
$\{\varphi^{(e)}\}$, strain matric
$[N_1\ N_2\ ...\ N_n\ ]$, shape function matric
$\{\Phi^{(e)}\}$, displacement matric
[k], stiffness matric local
$[K^{(e)}\ ]$, stiffness matric global

Appendix III (Example of exact analysis)

On (1,2375 x 10⁻² m, 0 m)

$$D = G\frac{\pi a^3 b^3}{a^2 + b^2}$$

$$= 8{,}02 \times 10^{11}\ (\mathrm{Nm^{-2}})$$

$$\times \frac{\pi\left((1{,}2375\times10^{-2}(\mathrm{m})\right)^3 \times \left(1{,}05\times10^{-2}(\mathrm{m})\right)^3}{\left(1{,}2375\times10^{-2}(\mathrm{m})\right)^2 + \left(1{,}05\times10^{-2}(\mathrm{m})\right)^2}$$

$$= 2{,}0985 \times 10^4\ \mathrm{Nm^2}$$

$$\beta = \frac{T}{D}$$

$$= \frac{2{,}5\ (\mathrm{N.m})}{2{,}0985\times10^4\ (\mathrm{Nm^2})}$$

$$= 1{,}19128 \times 10^{-4}\ (\mathrm{rad/m})$$

$$\tau = \frac{2G\beta ab}{a^2+b^2}\left(\frac{b^2x^2}{a^2} + \frac{a^2y^2}{b^2}\right)^{1/2} \quad \tau = 2\times 8{,}02\times10^{11}(\mathrm{N/m^2}) \times 1{,}19128\times10^{-4}\ (\mathrm{rad/m}) \times$$

$$= \frac{1{,}2375\times10^{-2}\ (\mathrm{m}) \times \left(1{,}05\times10^{-2}\ (\mathrm{m})\right)^2}{\left(1{,}2375\times10^{-2}\ (\mathrm{m})\right)^2 + \left(1{,}05\times10^{-2}\ (\mathrm{m})\right)^2}$$

$$= 9{,}89786 \times 10^{-1}\ (\mathrm{MPa})$$

Appendix IV (Numeric result)

ne	*nd*	τ_{yz}	τ_{xz}
381	1	-2.070794E-02	1.554892E-01
381	2	-9.1 63767E-02	1.37111 7E-01
381	3	-1.621277E-01	1.386381E-01
381	4	-1.975473E-02	1.524231E-01
:	:	:	:

381	25	-1 .715979E-02	-3.403563E-02
381	26	-7.59281 2E-02	-3.491 694E-02
381	27	-1.343517E-01	-3.201585E-02
382	1	-2.023174E-01	1.575184E-01
382	2	-2.725661 E-01	1.23701 7E-01

Appendix IV: Continue

382	3	-3.41 5524E-01	1.082025E-01
382	4	-1.92951 3E-01	1.49081 7E-01
:	:	:	:
382	25	-1 .686096E-01	-3.264806E-02
382	26	-2.258456E-01	-3.151811E-02
382	27	-2.820741 E-01	-2.694297E-02
383	1	-3.782694E-01	1.450308E-01
383	2	-4.465422E-01	9.829475E-02
383	3	-5.128767E-01	6.657960E-02
383	4	-3.608940E-01	1.31 6303E-01
:	:	:	:
383	25	-3.160669E-01	-2.795371 E-02
383	26	-3.7001 08E-01	-2.5091 67E-02
383	27	-4.223581 E-01	-1 .945557E-02
384	1	-5.444156E-01	1.193671E-01
384	2	-6.092029E-01	6.344362E-02
384	3	-6.715754E-01	1.735057E-02
:	:	:	:
384	25	-4.556626E-01	-2.044872E-02
384	26	-5.047951 E-01	-1.630881 E-02
384	27	-5.518476E-01	-1.039213E-02
385	1	-6.969805E-01	8.306074E-02
385	2	-7.565399E-01	2.2751 74E-02
385	3	-8.134387E-01	-3.487854E-02
385	4	-6.658669E-01	6.271 353E-02
:	:	:	:
385	25	-5.837399E-01	-1 .100087E-02
385	26	-6.268578E-01	-6.059848E-03
385	27	-6.675355E-01	-6.399453E-04
386	1	-8.325779E-01	3.932636E-02
386	2	-8.850116E-01	-1.958460E-02
386	3	-9.34751 0E-01	-8.470376E-02
386	4	-7.960476E-01	1.793592E-02
:	:	:	:
386	25	-6.969939E-01	-6.809568E-04
386	26	-7.332356E-01	4.656347E-03
386	27	-7.667969E-01	8.935000E-03
387	1	-9.481711E-01	-8.267000E-03
387	2	-9.91 5884E-01	-5.926113E-02
387	3	-1 .032429E+00	-1 .263809E-01
387	4	-9.072740E-01	-2.844540E-02
387	5	-9.492475E-01	-5.665671 E-02
387	6	-9.88531 7E-01	-1.02511 3E-01