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An Exact Solution of Perturbed Solitary Waves Due to KDv Equation

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Abstract: This paper obtains the exact 1-soliton solution of the perturbed Korteweg-de Vries equation with power law nonlinearity. The solitary wave ansatz is used to carry out this integration. The domain restrictions are identied in the process and the parameter constraints are also obtained. Finally, the numerical simulations are implemented in the paper.

Key words: solitary waves; integrability; perturbation.

INTRODUCTION

The Korteweg-de Vries (KdV) equation arises in the study of shallow water waves in the context of uid dynamics. This equation was first derived in 1895 by D.J. Korteweg and G. de Vries to model water waves in a shallow canal. Their goal was to settle a long-standing question: whether a solitary wave could persist under those conditions. Based on his personal observation of such waves since the 1830s, the naturalist John Scott Russell insisted that such waves do occur, but several prominent mathematicians, including Stokes, were convinced that they were impossible.

Korteweg and de Vries proved that Russell was correct by finding explicit, closed-form, traveling wave solutions to their equation that decay rapidly and thus it represents a highly localized moving hump. Both, the fact that such a solution to a nonlinear equation could exist and the fact that one could write it explicitly were later to be recognized as extremely important, but they went relatively unnoticed at the time.

The KdV equation did not receive signicant further attention until 1965, when N. Zabusky and M. Kruskal published the results of their numerical experimentation with the equation. Their computer generated approximate solutions to the KdV equation indicated that any localized initial prole that was allowed to evolve according to KdV equation eventually consisted of a nite set of localized traveling waves of the same shape as the original solitary waves discovered in 1895. Furthermore, when two of the localized disturbances collided, they would emerge from the collision again as an-other pair of traveling waves with a shift in phase as the only consequence of their interaction. Since the solitary waves made up these solutions seemed to behave like particles, Zabusky and Kruskal coined the term soliton to describe them.

Shortly after, another remarkable discovery was made concerning the KdV equation. It is possible to write many exact solutions to the KdV equation by using ideas from Inverse Scattering Transform (IST). In particular, the exact solution that is discussed in this paper is derivable from IST. In modern terminology, it can be said that this is the discovery of the first integrable nonlinear partial dierential equation (Antonova and Biswas, 2009; Biswas and Zerrad, 2008; Chen *et al.,* 2007; Feng, 2002; Osborne, 1997; Wazwaz, 2004; 2008; 2009; Zhang, 2007; Zhidkov, 2001).

MATHEMATICAL ANALYSIS

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The KdV equation with power law nonlinearity is given by

$$q_t + aq^n q_x + bq_{xxx} = 0 \tag{1}$$

Here in (1), *a* and b are constants. They respectively represent the coecient of nonlinear and dispersion terms. The first term is the evolution term, and thus these equations, mathematically, fall into the category of nonlinear evolution equations. The index n in the nonlinear term is the index of nonlinearity and is therefore called the power law nonlinearity. The special case where $n = 1$, (1) is the KdV equation and when $n = 2$, equation (1) is known as the modied KdV (mKdV) equation. It needs to be noted that for the power law KdV equation it is necessary to have n= 4 for soliton solutions to exist. Solitons are the outcome of a delicate balance between dispersion and nonlinearity.

Besides this form of power law KdV equation, there are various other versions of this equation that arises in many areas of Physical Sciences. They are the cylindrical KdV equation, Gardners equation [2], K(mn) equation (Wazwaz, 2009) and many more. These equations have all been studied and they all have been integrated and exact solution has been obtained.

Equation (1) is not integrable by the method of Inverse Scattering Transform that classically integrates the special cases where $n = 1$ and $n = 2$. However, (1) supports solitary waves of the form

$$q(x,t) = \frac{A}{\cosh^{\frac{2}{n}}[B(x-\upsilon t)]} \tag{2}$$

where in (2) A represents the amplitude of the soliton while B is the inverse width of the soliton and these are related as

$$B = n\sqrt{\frac{aA^n}{2b(n+1)(n+2)}} \tag{3}$$

which requires the condition

$$ab > 0 \tag{4}$$

to hold for bright solitons of the form given by (2) to exist. The velocity v of the soliton is given by

$$\upsilon = \frac{AbB^2}{n^2} \tag{5}$$

This solution is also known as the 1-soliton solution.

The perturbed KdV equation with power law nonlinearity is given by

$$\begin{aligned} &qt + aq^n q_x + bq_{xxx} \\ &= \gamma q_x q_{xx} + \delta q^2 q_x + \lambda q q_{xxx} + \nu q q_x q_{xx} \\ &+ \sigma q_x^3 + \xi q_x q_{xxx} + \theta q_{xx} q_{xxx} + \kappa q q_{xxxxx} \end{aligned} \tag{6}$$

In the perturbation terms, the coecient of is the higher order nonlinear dispersion In (5) the term with the coecient of will provide the higher stabilizing term and must therefore be taken into account. The remaining coecients appear in the context of Whitham hierarchy (Osborne 1997). In this paper, an exact 1-soliton solution to (6) will be obtained. It is interesting to note that (6). It is interesting to note that (6) is integrable in closed form (Wazwaz 2004, 2008) for the choice = 20, = 30 with m = 2, = 10, while = = = = = 0. This case is known as the Kodama equation (Antonova and Biswas, 2009; Osborne, 1997).

Non-topological Soliton Solution:

At the beginning of this section we assume that n > 2. In order to obtain an exact 1-soliton solution to (5), the following hypothesis is used as a starting point. It is assumed that the 1-soliton solution to (5) is given in the form (Antonova and Biswas, 2009):

$$q(x,t) = \frac{A}{\cosh^p \tau} \tag{7}$$

where the unknown exponent p will be obtained in course of derivation of the exact solution. Here

$$\tau = B(x\text{-}vt) \tag{8}$$

The solitons of the type given by (7) are known as non-topological solitons. It is justied to use this ansatz as (6) is a perturbed version of (1). The only dierence that will happen is that the variation of the amplitude and width of the soliton will now depend on the perturbation parameters and so is the velocity of the soliton. But the structural integrity of the soliton solution should stay the same as the unperturbed equation (1). The exact variation of these parameters will be obtained by using this solitary wave ansatz in the perturbed KdV equation. Thus, substituting this ansatz into (6) yields the following relation

$$\begin{aligned}
&\frac{p\upsilon AB}{\cosh^{p}\tau}-\frac{apA^{n+1}B}{\cosh^{(n+1)p}\tau}-\frac{bp^{3}AB^{3}}{\cosh^{p}\tau}+\frac{bp(p+1)(p+2)AB^{3}}{\cosh^{p+2}\tau}\\
&=\frac{p^{3}A^{2}B^{3}}{\cosh^{2p}\tau}+\frac{\gamma p^{2}(p+1)A^{2}B^{3}}{\cosh^{2p+2}\tau}-\frac{\delta pA^{3}B}{\cosh^{3p}\tau}\\
&-\frac{\lambda p^{3}A^{2}B^{3}}{\cosh^{2p}\tau}+\frac{\lambda p(p+1)(p+2)A^{2}B^{3}}{\cosh^{2p+2}\tau}-\frac{\nu p^{3}A^{3}B^{3}}{\cosh^{3p}\tau}\\
&+\frac{\nu p^{2}(p+1)A^{3}B^{3}}{\cosh^{3p+2}\tau}-\frac{\sigma p^{3}A^{3}B^{3}}{\cosh^{3p}\tau}+\frac{\sigma p^{3}A^{3}B^{3}}{\cosh^{3p+2}\tau}-\frac{\xi p^{3}A^{2}B^{5}}{\cosh^{2p}\tau}\\
&+\frac{2\xi p^{2}(p+1)(p^{2}2p+2)A^{2}B^{5}}{\cosh^{2p+2}\tau}-\frac{\xi p^{2}(p+1)(p+2)(p+3)A^{2}B^{5}}{\cosh^{2p+4}\tau}\\
&-\frac{\theta p^{2}(p+1)(p+2)A^{2}B^{5}}{\cosh^{2p+4}\tau}-\frac{\kappa p^{5}A^{2}B^{5}}{\cosh^{2p}\tau}\\
&+\frac{2\kappa p(p+1)^{2}(p+2)(p^{2}+2p+2)A^{2}B^{5}}{\cosh^{2p+2}\tau}\\
&-\frac{\kappa p(p+1)(p+2)(p+3)(p+4)A^{2}B^{5}}{\cosh^{2p+4}\tau}
\end{aligned} \tag{9}$$

Now, from (9) equating the exponents $(n + 1)$p and $p + 2$ yields

$$(n+1)p = p+2 \tag{10}$$

which gives

$$p = \frac{2}{n} \tag{11}$$

It needs to be noted that (9) has the linearly independent functions given by $1/\cosh^{p+j}\tau$ for $j = 0; 2;1/\cosh^{2p+j\tau}$ for $j = 0; 2; 4$ and $1/\cosh^{3p+j\tau}$ for $j = 0; 2$. So, setting their respective coe cients to zero yields (5), (3) as well as

$$B = \frac{n}{2}\sqrt{\frac{\gamma+\lambda}{\xi+\kappa+\theta}} \tag{12}$$

$$B = \frac{n}{2}\sqrt{\frac{\gamma+(n+1)\lambda}{(n+2)\theta+(n^{2}+2n+2)(\xi+(n+1)\kappa)}} \tag{13}$$

$$(3n+2)\xi+(n+2)\theta+(3n+2)(2n+1)\kappa = 0 \tag{14}$$

$$B = \sqrt{\frac{n\delta}{2\nu}} \tag{15}$$

and

$$(n+2)\nu+2\sigma = 0 \tag{16}$$

Now equations (12), (13) and (15) respectively imposes the restrictions

$$(\xi+\kappa+\theta)(\gamma+\lambda)<0 \tag{17}$$

$$\{\gamma+(n+1)\lambda\}\left[(n+2)\theta+(n^2+2n+2)\{\xi+(n+1)\kappa\}\right]<0 \tag{18}$$

$$\delta\nu<0 \tag{19}$$

Now, equating the three values of the inverse width B pairwise, yields the following three additional constraints on the perturbation coefficients

$$2\delta(\xi+\kappa+\theta)+n\nu(\gamma+\lambda)=0 \tag{20}$$

$$n\nu\{\gamma+(n+1)\lambda\}=2\delta\left[(n+2)\theta+(n^2+2n+2)\{\xi+(n+1)\kappa\}\right] \tag{21}$$

$$(\xi+\kappa+\theta)\{\gamma+(n+1)\lambda\}=(\gamma+\lambda)\left[(n+2)\theta+(n^2+2n+2)(\xi+(n+1)\kappa)\right] \tag{22}$$

Finally, the velocity of the soliton from (5) can be rewritten, after substituting the value of B from (12), (13) and (15), in the following three equivalent forms

$$\nu=\frac{(\gamma+\lambda)b}{\xi+\kappa+\theta} \tag{23}$$

$$\nu=\frac{\{\gamma+(n+1)\lambda\}b}{(n+2)\theta+(n^2+2n+2)(\xi+(n+1)\kappa)} \tag{24}$$

and

$$\nu=\frac{2\delta b}{n\nu} \tag{25}$$

Hence, finally, the 1-soliton solution to (6) is given by (2) where the amplitude-width relation is given by (3), while the width of the soliton is given by (12), (13) or (15) while the velocity is given by (23), (24) or (25). These imposes a number of constraints and parameter domain restrictions as seen. The special cases of KdV and mKdV equations easily follow from these relations by substituting n = 1 or n = 2 respectively. The following gure shows the prole of the KdV equation with n = 1 and a=b=1.

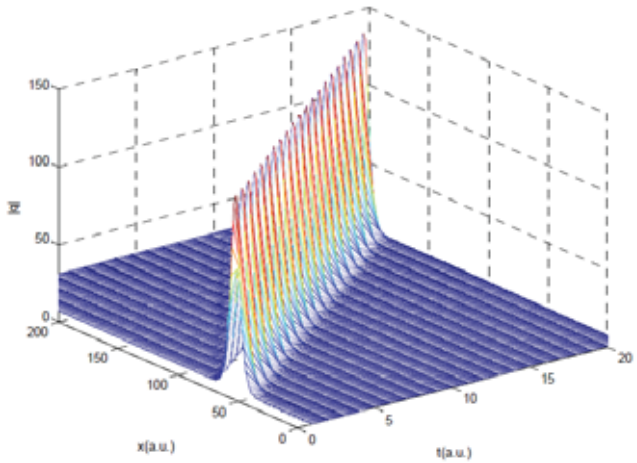

Fig. 1: 1-soliton solution of KdV equation for a=6, b=2, n=2.

Conclusions:

This paper integrates the perturbed KdV equation with power law nonlinearity, in presence of perturbation terms. An exact 1-soliton solution is obtained together with a number of parameter constraints and domain restrictions on the perturbation coecients.

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REFERENCES

Antonova, M., A. Biswas, 2009. Adiabatic parameter dynamics of perturbed solitary waves. Communications in Nonlinear Science and Numerical Simulation., 14(3): 734-748.

Biswas, A., E. Zerrad, 2008. Soliton perturbation theory for the Gardner equation. Advanced Studies in Theoretical Physics., 2(16): 787-794.

Chen, C.Y., J.R. Hsu, M.H. Cheng, H.H. Chen, C.F. Kuo, 2007. An investigation on internal solitary waves in a two layer uid: Propagation and reection from steep slopes. Ocean Engineering., 34(1): 171-184.

Feng, Z., 2002. Qualitative Analysis and Exact Solutions to the Burgers-Korteweg-de Vries Equation. Dynamics of Continuous, Discrete and Impulsive Systems, Series A., 9(4): 563-580.

Osborne, A.R., 1997. Approximate Asymptotic Integration of a higher order water-wave equation using the inverse scattering transform. Nonlinear Processes in Geophysics., 4(1): 29-53.

Wazwaz, A.M., 2004. Variants of the generalized KdV equation with compact and noncompact structures. Computers and Mathematics with Applications., 47(4-5): 583-591.

Wazwaz, A.M., 2008. New sets of solitary wave solutions to the KdV, mKdV and the generalized KdV equations. Communications in Nonlinear Science and Numerical Simulation., 13(2): 331-339.

Wazwaz, A.M., 2009. Bright and dark soliton solutions for a K(mn) equation with t-dependent coecients. Physics Letters A. 373(25): 2162-2165.

Zhang, S., 2007. Exact solution of a KdV equation with variable coecients via exp-function method. Nonlinear Dynamics., 52(1-2): 11-17.

Zhidkov, P.E., 2001. Korteweg-de Vries and Nonlinear Schrodingers Equations: Qualitative Theory. Springer Verlag. New York.