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Numerical Solution of Brusselator Model by Expansion Methods

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Abstract: In this paper, four types of weighted residual methods (Collocation, Subdomain, Galerkin and least-square methods) are presented for finding an approximate solution of the Brusselator model. We showed the efficiency of the prescribed methods by solving numerical example.

Key words: Collocation method, Subdomain method, Galerkin methods, least-square method, Brusselator model

INTRODUCTION

One of the uses of approximating functions is to replace complicated functions by some simpler functions so that many operations such as integration can be easily performed. In this paper, a solution to the Brosselator system:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = d_1\frac{\partial^2 u_1}{\partial t^2} + u_1^2u_2 - (b+1)u_1 + a, \quad (t,x) \in (0,\infty)\times\Omega = D,\\
&\frac{\partial u_2}{\partial t} = d_2\frac{\partial^2 u_2}{\partial t^2} - u_1^2u_2 + bu_1, \qquad (t,x) \in (0,\infty)\times\Omega = D\\
&\quad u_1(t,x) = u_2(t,x) = 0, t > 0, x \in \partial\Omega\\
&\quad u_1(0,x) = u_{10}(x), \quad u_2(0,x) = u_{20}(x), \quad x \in \Omega,
\end{aligned} \tag{1}$$

is approximated by polynomials in two independent variables. In two dimensions, a complete n^{th} degree polynomial is given by

$$u_i(x,t) » P_{in}(x,t) = \sum_{\substack{j=0}}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{ijk}\, x^j t^k \quad, \ i=1, 2,...,n \tag{2}$$

where j and k are permuted accordingly. The number of terms in the above polynomials is equal to

$$\frac{(n+1)\times(n+2)}{2}\cdot$$

For example:

if n=1,

$$P_{i1}(x,t) = c_{i00} + c_{i01}t + c_{i10}x \quad , \text{(3 terms)}$$

if n=2,

$$P_{i2}(x,t) = c_{i00} + c_{i01}t + c_{i02}t^2 + c_{i10}x + c_{i11}xt + c_{i20}x^2 \ , \text{(6 terms)}$$

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if n=3,

$$P_{i3}(x,t) = c_{i00} + c_{i01}t + c_{i02}t^2 + c_{i03}t^3 + c_{i10}x + c_{i11}xt + c_{i12}xt^2 + c_{i20}x^2 + c_{i21}x^2t + c_{i30}x^3 \text{ . (10 terms)}$$

In order to consolidate the expansion method, some error minimizing technique to determine the coefficients $\underset{j+k\le n}{c_{ijk}}$, i=1, 2,…,n; j, k=0, 1,...,n in (1) are needed, one of the most popular minimizing techniques is the weighted residual method (WRM) (Jain (1979), Saeed (2006)) which include the {collocation method (CM), partition method (PM), Galerkin method (GM) and Least-square method(LSM)}. In this chapter, we choose the weighted residual methods for finding the unknown coefficient.

Formulation of WRM's for Solving Brusselator System:

In this section, we formulate WRM's to be suitable for solving Brusselator system as follows: Consider the functional equation given by

$$T(u_i(x,t)) = p(x) \quad (x,t) \in D \quad , i=1, 2,\dots, n. \tag{3}$$

Now by substituting the approximated solution of $u_i(x,t)$ given by (2) into the equation (3), the result is so called "residual function' defined by:

$$R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t) = T(u_i(x,t)) - p(x) \quad , \tag{4}$$

where i=1, 2, …, n; j, k=0, 1,…, n.

The residue $R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)$ depends on x, t as well as the way that the parameters $\underset{j+k\le n}{c_{ijk}}$, i=1, 2, …, n; j, k=0, 1,…, n are chosen. Since the residual function $R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)$ is identically equal to zero for the exact solution, the goal is to choose the coefficients $\underset{j+k\le n}{c_{ijk}}$ so that the residual function can be minimized.

The main a im for using the *WRM*'s is to find the coefficients $\underset{j+k\le n}{c_{ijk}}$ so that the residue $R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)$ becomes small (in fact zero) over a chosen domain. In integral form this can be achieved with the condition

$$\iint_D w_{jk}(x,t) R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)\,dt\,dx = 0 \tag{5}$$

where $w_{jk}(x,t)$ are prescribed weight function, the technique described by (5) is called *WRM*'s, by which the optimal values of $L=\dfrac{n\times(n+1)\times(n+2)}{2}$ coefficients $\underset{j+k\le n}{c_{ijk}}$ that minimize $R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)$, is obtained.

We now presented four methods of the *WRM*'s to determine $\underset{j+k\le n}{c_{ijk}}$ in the equation (2) as follows:

Collocation Method (CM):

The main idea behind this method is the parameters $\underset{j+k\le n}{c_{ijk}}$, i=1, 2,…,n; j, k=0, 1,...,n that are to be found by requiring that the residual $R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)$ be zero at the given sets of the collocation points $x_0, x_1, \dots, x_n$

and $t_0, t_1, ..., t_n$ on the interval [a_1, b_1] and [a_2,b_2] respectively, where $x_j = a_1 + jh_x$, $t_k = a_2 + kh_t$; j, k=0, 1,...,n $h_x = \frac{b_1 - a_1}{n}$ and $h_t = \frac{b_2 - a_2}{n}$ as follows:

In this method the weighted functions will be Dirac delta functions in two dimensional Cartesian coordinates (Hassani (2000)), i.e.

$$w_{jk}(x,t) = \delta^2(x,t) = \delta(x - x_j)\delta(t - t_k) \quad ,$$

which vanishes everywhere except at $x = x_j$, $j = 0,1,...,n$ and $t = t_k$, $k = 0,1,...,n$ such that $j + k \le n$. This means that

$$\delta(x - x_j) = \begin{cases} 0 & \text{if } x \ne x_j \\ 1 & \text{if } x = x_j \end{cases} \quad \text{for } j = 0, 1, ..., n$$

and

$$\delta(t - t_k) = \begin{cases} 0 & \text{if } t \ne t_k \\ 1 & \text{if } t = t_k \end{cases} \quad \text{for } k = 0, 1, ..., n.$$

Then the equation (5), becomes

$$\iint_D w_{jk}(x,t) R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)dtdx = \iint_D \delta(x - x_j)\delta(t - t_k) R_{in}[\underset{j+k\le n}{c_{ijk}}, x, t]dtdx = 0 \quad ,$$

this can be written as

$$R_{in}(\underset{j+k\le n}{c_{ijk}}, x_r, t_q) = 0 \quad , i = 1, 2, ..., n; \; r, q = 0, 1, ..., n \text{ such that } r + q \le n. \tag{6}$$

From equation (6), we get an $L \times L$ nonlinear system of simultaneous equations for the coefficients $\underset{j+k\le n}{c_{ijk}}$, where $L = \frac{n \times (n+1) \times (n+2)}{2}$, which can be solved by modified Newton-Raphson method for finding the parameters $\underset{j+k\le n}{c_{ijk}}$.

Sub-Domain Method (SDM):

The idea in this section is to force the weighted residual equal to zero not just at fixed points in the domain, but over various subsections of the domain. To accomplish this, the weight functions are set to unity and the integral over the entire domain is broken into a number of sub-domains sufficient to evaluate all unknown parameters $\underset{j+k\le n}{c_{ijk}}$ as follows:

First, divide each of the intervals [a_1, b_1] and [a_2, b_2] into n+1 sub-intervals D_j=[x_j,x_{j+1}] and D_k=[t_k,t_{k+1}] respectively where x_j=a_1+jh_x, t_k=a_2+kh_t; j, k=0, 1,...,n, $h_x = \frac{b_1 - a_1}{n+1}$, $h_t = \frac{b_2 - a_2}{n+1}$ and

$$w_{jk}(x,t) \;=\begin{cases}1 & if \;\; x\in D_j \;\; and \;\;\; t\in D_k \\ 0 & if \;\;\; x\notin D_j \;\; or \; t\notin D_k\end{cases} \quad ; j,\ k{=}0,\ 1,\ldots,\ n$$

Second, integrate the residual (4) with respect to x and t such that for each sub-intervals D_j we must use all sub-intervals D_k provided that the sum of indexes of the lower limits of the integrals less than or equal to N. Hence the equation (5) becomes

$$\int_{x_j}^{x_{j+1}}\int_{t_k}^{t_{k+1}} R_{in}(\underset{j+k\le n}{c_{ijk}}, x,t)dtdx = \;\; 0,\ i{=}1,\ 2,\ldots,n;\ j,\ k{=}0,\ 1,\ldots,n \text{ such that } j+k\le n$$

This means that:

$$\int_{x_0}^{x_1}\int_{t_v}^{t_{v+1}} R_{in}(\underset{j+k\le n}{c_{ijk}}, x,t)dtdx = \;\; 0, \text{ for } i{=}1,\ 2,\ \ldots,\ n \text{ and } v{=}0,\ 1,\ldots,\text{n}$$

$$\int_{x_1}^{x_2}\int_{t_v}^{t_{v+1}} R_{in}(\underset{j+k\le n}{c_{ijk}}, x,t)dtdx = \;\; 0, \text{ for } i{=}1,\ 2,\ \ldots,\ n \text{ and } v{=}0,\ 1,\ldots,\text{n-1}$$

$$\int_{x_2}^{x_3}\int_{t_v}^{t_{v+1}} R_{in}(\underset{j+k\le n}{c_{ijk}}, x,t)dtdx = \;\; 0, \text{ for } i{=}1,\ 2,\ \ldots,\ n \text{ and } v{=}0,\ 1,\ldots,\text{n-2} \qquad (7)$$

$$\int_{x_N}^{x_{N+1}}\int_{t_v}^{t_{v+1}} R_{in}(\underset{j+k\le n}{c_{ijk}}, x,t)dtdx = \;\; 0, \text{ for } i{=}1,\ 2,\ \ldots,\ n \text{ and } v{=}0.$$

Equation (7), yields an $L\times L$ nonlinear system of simultaneous equations for the coefficients $\underset{j+k\le n}{c_{ijk}}$. Solve this system by Modified Newton-Raphson method for finding the parameters $\underset{j+k\le n}{c_{ijk}}$.

Least Square Method (LSM):

In this method, the continuous summation of all the squared residual is minimized; the rationale behind the name can be seen. In other words, a minimum of

$$S{=}\iint_D R_{in}^2(\underset{j+k\le n}{c_{ijk}}, x,\ t)\ dtdx \ .$$

in order to achieve a minimum of this scalar function, the derivatives of S with respect to all the unknown parameters must be zero. That is

$$\frac{\partial S}{\partial \underset{r+q\le n}{c_{irq}}} = 0 = 2 \quad \iint_D R_{in}(\underset{j+k\le n}{c_{ijk}}, x,\ t)\frac{\partial R_{in}(\underset{j+k\le n}{c_{ijk}}, x,\ t)}{\partial \underset{r+q\le n}{c_{irq}}}\ dtdx \qquad (8)$$

for i=1, 2, …, n; r=0, 1, …, n and q=0, 1, …, n such that $r+q\le n$.

Comparing with (5), the weight functions are to be

$$w_{irq}(x,t) = 2 \quad \frac{\partial R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)}{\underset{r+q\le n}{\partial c_{irq}}} \quad .$$

Hence, from the equation (8) we get

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t) \frac{\partial R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t)}{\underset{r+q\le n}{\partial c_{irq}}} dtdx = 0, \tag{9}$$

for i=1, 2, …, n; r=0, 1,…, n and q=0, 1,…, n such that $r + q \le n$.
The equation (9) is also a system of L nonlinear equations in the L unknown coefficients $\underset{j+k\le n}{c_{ijk}}$. Solve this system by modified Newton-Raphson method for finding the parameters $\underset{j+k\le n}{c_{ijk}}$.

Galerkin Method (GM):

This method may be viewed as a modification of the LSM, rather than using the derivative of the residual with respect to the unknown $\underset{j+k\le n}{c_{ijk}}$, the derivative of the approximating function (2) is used. That is, if the function is approximated as in (2), then the weight functions are

$$w_{irq}(x,t) = \frac{\partial P_{in}(x,t)}{\partial \underset{r+q\le n}{c_{irq}}} \text{ , } i=1, 2,\dots,n \text{ and } r, q=0, 1,\dots,n \text{ such that } r + q \le n$$

Hence, from the equation (3.8) we get

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} R_{in}(\underset{j+k\le n}{c_{ijk}}, x, t) \frac{\partial P_{in}(x, t)}{\underset{r+q\le n}{\partial c_{irq}}} dtdx = 0, \tag{10}$$

for i=1, 2, …, n; and r, q=0, 1,…, n such that $r + q \le n$

The above system, is also $L \quad L$ nonlinear system and we solved it by modified Newton-Raphson method for finding the parameters $\underset{j+k\le n}{c_{ijk}}$.

Solution of the Brusselator System by Using WRM's:

In this section, we approximate the solution of Brusselator system (1) by means of the *WRM*'s, described in section 2 and we attempt to calculate the coefficients $\underset{j+k\le n}{c_{ijk}}$, i=1, 2,…,n; j, k=0, 1,…,n as follows:

Using (1) and (3), we can write the residual equation (3) in the form

$$R_{1n}(\underset{j+k\le n}{c_{1jk}},x,t)= d_1\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t^k\right) - \frac{\partial}{\partial t}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t^k\right) -$$

$$\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t^k\right)^2\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x^j t^k\right) +(b+1)\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t^k - a \quad ,$$

$$R_{2n}(\underset{j+k\le n}{c_{2jk}},x,t)= \frac{\partial}{\partial t}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x^j t^k\right) - d_2\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x^j t^k\right) \qquad (11)$$

$$+\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t^k\right)^2\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x^j t^k\right) - b\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t^k \quad .$$

Hereunder, we try to find the parameters $\underset{j+k\le n}{c_{ijk}}$, i=1,2 using the methods described in section 2 as follows:

Collocation Method:

From the equations (11) and (6) we get

$$\frac{\partial}{\partial t}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_o^j t^k\right)\Bigg|_{t=t_q} - d_1\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t_q^k\right)\Bigg|_{x=x_0} - \left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_0^j t_q^k\right)^2\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x_0^j t_q^k\right)$$

$$+(b+1)\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_0^j t_q^k = a \quad , \quad \frac{\partial}{\partial t}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x_0^j t^k\right)\Bigg|_{t=t_q} - d_2\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x^j t_q^k\right)\Bigg|_{x=x_0}$$

$$+\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_0^j t_q^k\right)^2\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x_0^j t_q^k\right) - b\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_0^j t_q^k = 0 \quad \text{, for } q=0,\,1,\,\ldots,\,n \qquad (12)$$

$$\frac{\partial}{\partial t}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_1^j t^k\right)\Bigg|_{t=t_q} \quad d_1\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x^j t_q^k\right)\Bigg|_{x=x_1} - \left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{1jk}\,x_1^j t_q^k\right)^2\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x_1^j t_q^k\right)$$

$$+(b+1)\sum_{j=0}^{N}\sum_{\substack{k=0\\ j+k\le n}}^{N}c_{1jk}\,x_1^j t_q^k = a \quad , \quad \frac{\partial}{\partial t}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x_1^j t^k\right)\Bigg|_{t=t_q} - d_2\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n}c_{2jk}\,x^j t_q^k\right)\Bigg|_{x=x_1}$$

$$+\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x_1^j t_q^k\right)^2\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x_1^j t_q^k\right)\quad -b\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x_1^j t_q^k=0\text{ , for } q=0,\ 1,\ \ldots,\ n\text{-}1$$

$$\vdots$$

$$\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x_n^j t^k\right)\Bigg|_{t=t_q}\quad -d_2\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t_q^k\right)\Bigg|_{x=x_n}$$

$$+\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x_n^j t_q^k\right)^2\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x_n^j t_q^k\right)\quad -b\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x_n^j t_q^k=0\text{ , for } q=0.$$

The equation (12) is a nonlinear system of L nonlinear equations in L unknowns $\underset{j+k\le n}{c_{ijk}}$; i=1, 2.

Solve the non linear system using modified Newton-Raphson method; to find the unknowns and substitute the value of $\underset{j+k\le n}{c_{ijk}}$; i=1, 2 in the equation (2), we get the approximate solution of (bruss.system chapter one).

Sub-Domain Method:

From the equation (11) and (7) we get

$$\int_{x_0}^{x_1}\int_{t_0}^{t_{q+1}}\left\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)\quad -d_1\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)\quad \left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)^2\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right)\right.$$

$$\left.+(b+1)\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right\}dtdx=a\int_{x_0}^{x_1}\int_{t_0}^{t_{q+1}}dtdx\quad ,\ \int_{x_0}^{x_1}\int_{t_0}^{t_{q+1}}\left\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right)\quad -d_2\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right)\right.$$

$$\left.+\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)^2\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right)\quad -b\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right\}dtdx=0\text{ , for } q=0,\ 1,\ \ldots,\ n$$

$$\int_{x_1}^{x_2}\int_{t_0}^{t_{q+1}}\left\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)\quad -d_1\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)\quad -\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)^2\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right)\right.$$

$$+ (b+1)\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k \Bigg\} dtdx = a\int_{x_1}^{x_2}\int_{t_0}^{t_{q+1}} dtdx \quad , \int_{x_1}^{x_2}\int_{t_0}^{t_{q+1}}\Bigg\{\frac{\partial}{\partial t}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad d_2\frac{\partial^2}{\partial x^2}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) -$$

$$+\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg)^2\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad - b\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg\} dtdx = 0 \quad \text{, for } q=0,\ 1,\ \ldots,\ n\text{-}1$$

$$\vdots$$

$$\int_{x_n}^{x_{n+1}}\int_{t_0}^{t_{q+1}}\Bigg\{\frac{\partial}{\partial t}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg) \quad d_1\frac{\partial^2}{\partial x^2}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg) \quad -\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg)^2\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad -$$

$$+ (b+1)\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k \Bigg\} dtdx = a\int_{x_n}^{x_{n+1}}\int_{t_0}^{t_{q+1}} dtdx \quad , \quad \int_{x_n}^{x_{n+1}}\int_{t_0}^{t_{q+1}}\Bigg\{\frac{\partial}{\partial t}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad - d_2\frac{\partial^2}{\partial x^2}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg)$$

$$+\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg)^2\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad - b\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg\} dtdx = 0 \quad \text{for } q=0.$$

The equation (13) is a linear system of L nonlinear equations in L unknowns $\underset{j+k\le n}{c_{ijk}}\ ; i=1,\ 2$. Solve the linear system using modifed Newton-Raphson method; to find the unknowns and substitute the value of $\underset{j+k\le n}{c_{ijk}}\ ; i=1,\ 2$ in the equation (2), we get the approximate solution of (bruss chapter one).

Least Square Method:

From the equation (11) and (9) we get

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Bigg\{\frac{\partial}{\partial t}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg) \quad - d_1\frac{\partial^2}{\partial x^2}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg)$$

$$\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg)^2\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad + (b+1)\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{1jk}\, x^j t^k\Bigg\}$$

$$\frac{\partial R_{1n}(\underset{j+k\le n}{c_{1jk}}\ , x,\ t)}{\underset{r+q\le n}{\partial c_{1rq}}} dtdx \qquad = a\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial R_{1n}(\underset{j+k\le n}{c_{1jk}}\ \ x,t)}{\underset{r+q\le n}{\partial c_{1rq}}} dtdx \quad , \tag{14}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Bigg\{\frac{\partial}{\partial t}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg) \quad - d_2\frac{\partial^2}{\partial x^2}\Bigg(\sum_{j=0}^{n}\sum_{\substack{k=0\\ j+k\le n}}^{n} c_{2jk}\, x^j t^k\Bigg)$$

$$+\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)^{2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^{j}t^{k}\right)\ -b\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\Bigg\}\ \frac{\partial R_{2n}(\underset{j+k\le n}{c_{2jk}},x,t)}{\underset{r+q\le n}{\partial c_{2rq}}}dtdx=0\ .$$

for i=1, 2, ..., n; r=0, 1,..., n and q=0, 1,..., n such that r + q $\le$ n.

The equation (14) is a linear system of L nonlinear equations in L unknowns $\underset{j+k\le n}{c_{ijk}}\ ;i=1,2$. Solve the linear system using modifed Newton-Raphson method; to find the unknowns and substitute the value of $\underset{j+k\le n}{c_{ijk}}\ ;i=1,2$ in the equation (2), we get the approximate solution of (bruss chapter one).

Galerkin Method:

From the equation (10) and (11) we get

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Bigg\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)\ -d_1\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)\ -$$

$$\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)^{2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^{j}t^{k}\right)\ +(b+1)\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\Bigg\}\ \frac{\partial P_{1n}(x,t)}{\underset{r+q\le n}{\partial c_{1rq}}}dtdx\ =a\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial P_{1n}(x,t)}{\underset{r+q\le n}{\partial c_{1rq}}}dtdx\ ,\qquad(15)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Bigg\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^{j}t^{k}\right)\ -d_2\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^{j}t^{k}\right)$$

$$+\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)^{2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^{j}t^{k}\right)\ -b\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\Bigg\}\ \frac{\partial P_{2n}(x,t)}{\underset{r+q\le n}{\partial c_{2rq}}}dtdx=0\ .$$

for i=1, 2, ..., n; r=0, 1,..., n and q=0, 1,..., n such that r + q $\le$ n.

After taking the derivatives of $P_{in}(x,t)$ with respect to $\underset{j+k\le n}{c_{ijk}}\ ;i=1,2$ in the equation (15) we get the following equation:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Bigg\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)\ -d_1\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)-$$

$$\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\right)^{2}\left(\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^{j}t^{k}\right)\ +(b+1)\sum_{\substack{j=0\\j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^{j}t^{k}\Bigg\}\ x^{r}t^{q}dtdx\ =a\int_{a_1}^{b_1}\int_{a_2}^{b_2}x^{r}t^{q}dtdx\ ,$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left\{\frac{\partial}{\partial t}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right) - d_2\frac{\partial^2}{\partial x^2}\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right)\right.$$

$$\left.+\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right)^2\left(\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{2jk}\,x^j t^k\right) - b\sum_{\substack{j=0\\ j+k\le n}}^{n}\sum_{k=0}^{n}c_{1jk}\,x^j t^k\right\} x^r t^q\,dtdx = 0\,.$$

for i=1, 2, …, n; r=0, 1,…, n and q=0, 1,…, n such that $r + q \le n$.

The equation (16) is a linear system of L nonlinear equations in L unknowns $\underset{j+k\le n}{c_{ijk}}\,;\, i=1,2$. Solve the linear system using modified Newton-Raphson method; to find the unknowns and substitute the value of $\underset{j+k\le n}{c_{ijk}}\,;\, i=1,2$ in the equation (2), we get the approximate solution of (1).

Practical Application:

Example 1:

$$\frac{\partial u}{\partial t} = d_1\frac{\partial^2 u}{\partial x^2} + \alpha + u^2 v - (B+1)u$$

$$\frac{\partial v}{\partial t} = d_2\frac{\partial^2 v}{\partial x^2} - u^2 v + Bu$$

$$u(0,t) = u(1,t) = \alpha, v(0,t) = v(1,t) = \frac{\beta}{\alpha}$$

$$u(x,0) = \alpha + x(1-x), v(x,0) = \frac{\beta}{\alpha} + x^2(1-x)$$

where $\alpha = 0.6,\ \beta = 0.2,\ d_1 = d_2 = 1/40.$

Tables (1) and (2) shows the numerical solution of u and v of Example 1 at time t=0.5.

Table 1: Numerical results of $u(x,t)$ by expansion method with t=0.5

$0 \le x \le 1$	CM	PM	GM	LSE
0	0.6000	0.6000	0.6000	0.6000
0.1	0.6473	0.6457	0.6452	0.6454
0.2	0.7173	0.7180	0.7182	0.7181
0.3	0.7673	0.7698	0.7706	0.7702
0.4	0.7973	0.8012	0.8023	0.8018
0.5	0.8073	0.8121	0.8135	0.8128
0.6	0.7973	0.8026	0.8039	0.8033
0.7	0.7673	0.7726	0.7738	0.7732
0.8	0.7173	0.7222	0.7230	0.7225
0.9	0.6473	0.6513	0.6517	0.6513
1	0.6000	0.6000	0.6000	0.6000

Table 2: Numerical results of v(x,t) by expansion method with t=0.5

$0 \le x \le 1$	CM	PM	GM	LSE
0	0.3333	0.3333	0.3333	0.3333
0.1	0.3277	0.3256	0.3271	0.3264
0.2	0.3977	0.3979	0.4000	0.3990
0.3	0.4477	0.4497	0.4523	0.4511

0.4	0.4777	0.4811	0.4840	0.4826
0.5	0.4877	0.4920	0.4951	0.4936
0.6	0.4777	0.4825	0.4856	0.4841
0.7	0.4477	0.4525	0.4555	0.4540
0.8	0.3977	0.4021	0.4048	0.4034
0.9	0.3277	0.3312	0.3335	0.3323
1	0.3333	0.3333	0.3333	0.3333

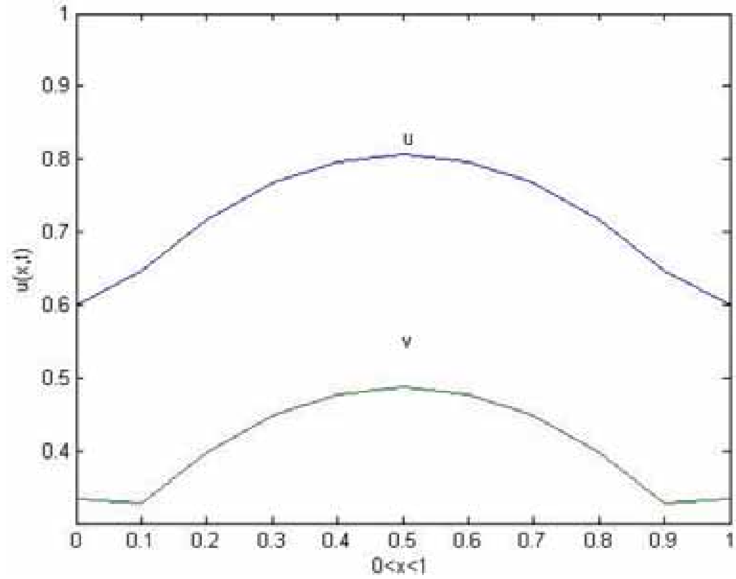


Fig. 1: Approximat solution by CM.

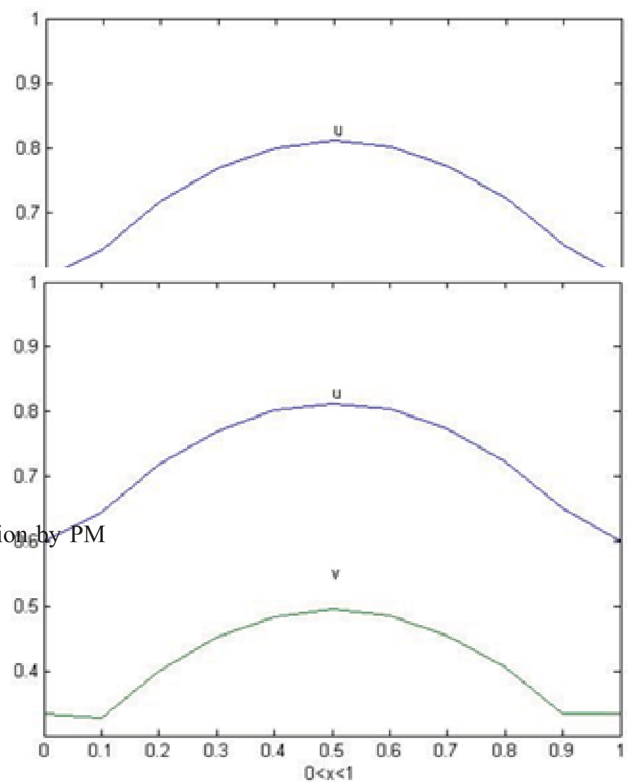


Fig. 2: Approximat solution by PM

Fig. 3: Approximat solution by GM

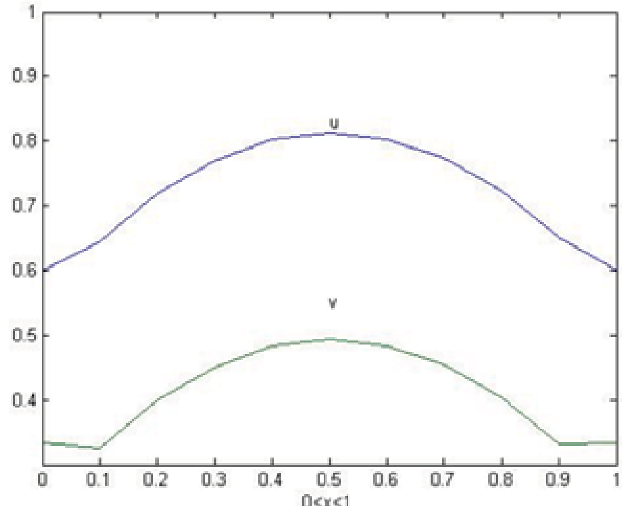
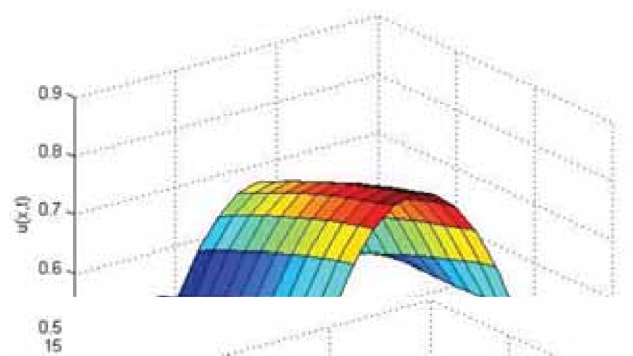


Fig. 4: Approximat solution by LSM



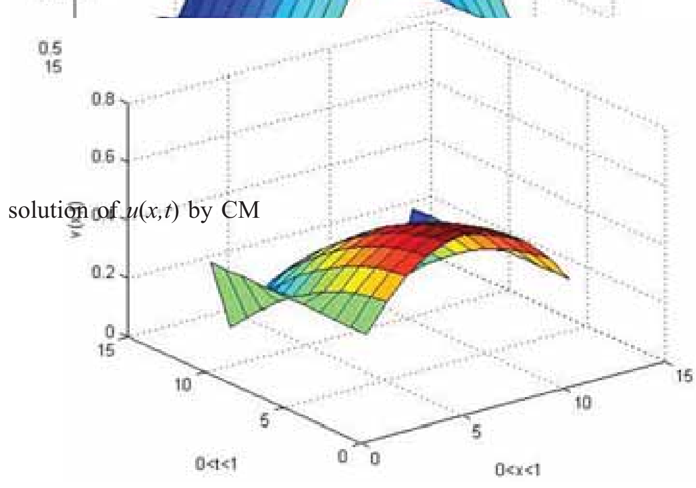


Fig. 5: Approximate solution of $u(x,t)$ by CM

Fig. 6: Approximate solution of $v(x,t)$ by CM

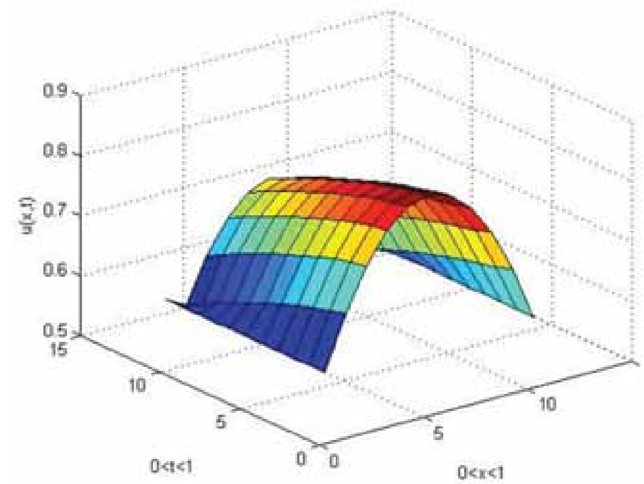


Fig. 7: Approximate solution of $u(x,t)$ by PM

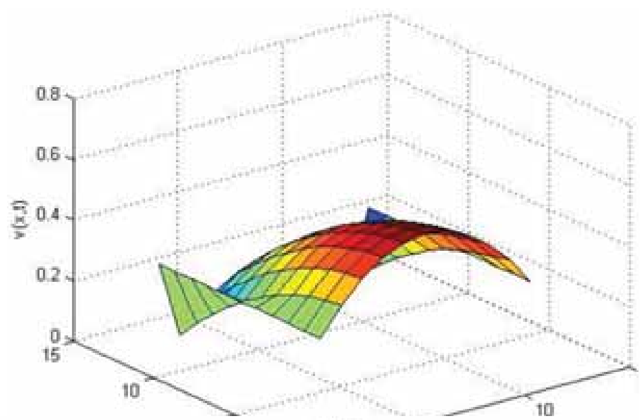


Fig. 8: Approximate solution of $v(x,t)$ by PM

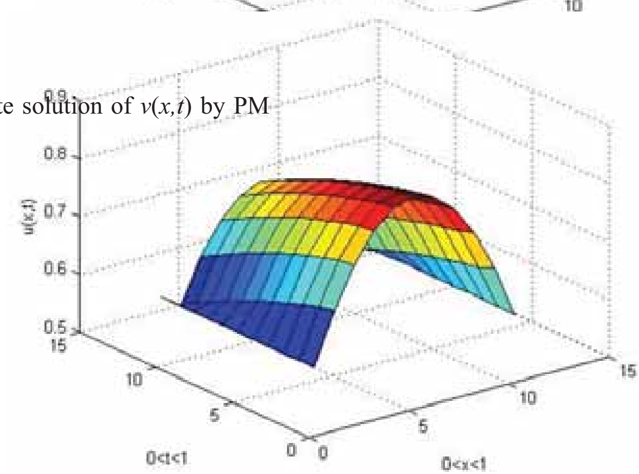


Fig. 9: Approximate solution of $u(x,t)$ by GM

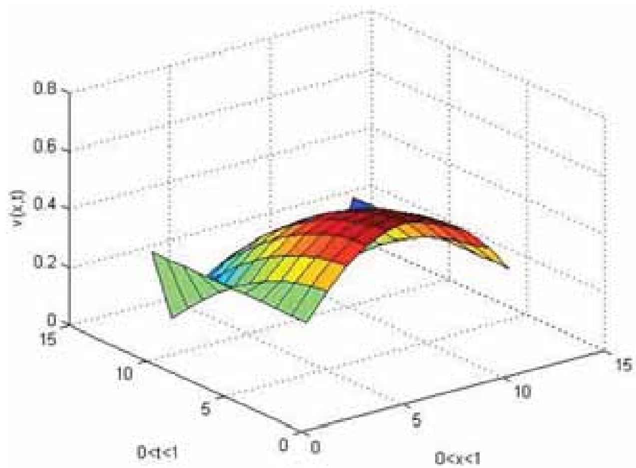


Fig. 10: Approximate solution of $v(x,t)$ by GM

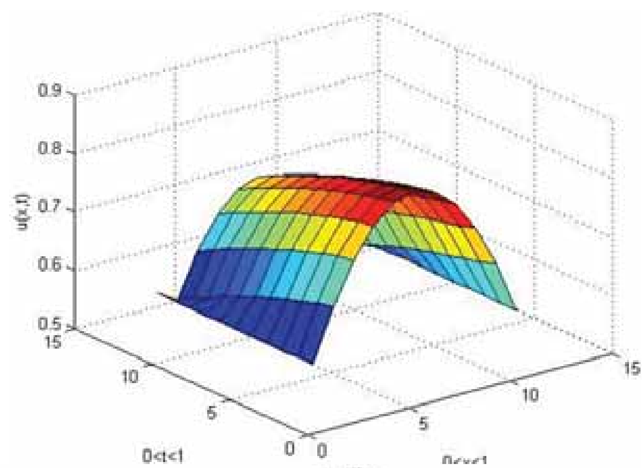


Fig. 11: Approximate solution of $u(x,t)$ by LSM

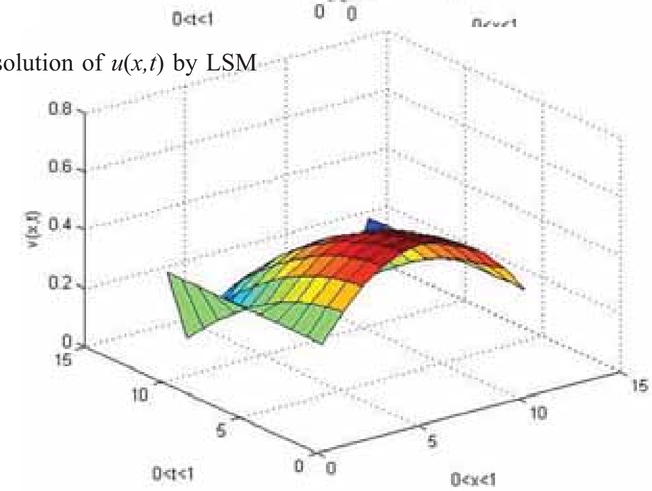


Fig. 12: Approximate solution of $v(x,t)$ by LSM

Conclusion:

We saw that CM, PM, GM and LSM gives approximately equal values as shown in Tables 1,2. Figures 1-12.

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