Solving Nonlinear Integral Equations of the Hammerstein-type by Using Double Exponential Transformation

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Abstract: In this paper, a Sinc-collocation method based on the double exponential transformation for solving Fredholm and Volterra Hammerstein integral equations is presented. Some properties of the Sinc-collocation method required for our subsequent development are given and utilized to reduce the computation of solution of the Hammerstein integral equations to some algebraic equations. Numerical examples are included to demonstrate the validity and applicability of the method. The method is easy to implement and yields very accurate results.

Key words: Nonlinear integral equations; Fredholm; Volterra; Hammerstein equations; Double exponential transformation.

INTRODUCTION

Consider the nonlinear integral equation of the form

\[ u(x) = f(x) + (KHu)(x), \quad x \in \Gamma = [a, b]. \]  

(1)

The Eq. (1) is Hammerstein integral equation of Volterra type in the form

\[ u(x) = f(x) + \int_{a}^{x} K(x, t)H(t, u(t))dt, \quad x \in \Gamma = [a, b], \]

(2)

and Fredholm type in the form

\[ u(x) = f(x) + \int_{\Gamma} K(x, t)H(t, u(t))dt, \quad x \in \Gamma = [a, b], \]

(3)

where a, b are constants, \( f(x), K(xt), H(tu(t)) \) are known functions and \( u(x) \) is a solution to be determined.

The Hammerstein integral equations have strong physical background, and arise from the electro-magnetic fluid dynamics. Moreover, Eq. (1) arises as a reformulation of two-point boundary value problems with a certain nonlinear boundary condition, (Delves, L.M., J.L. Mohamed, 1985; Atkinson, K.E., 1997). Also, multi-dimensional analogues of Eq. (1) appear as various reformulations of an elliptic partial differential equation with non-linear boundary conditions (Atkinson, K.E., 1994; Atkinson, K.E., G. Chandler, 1990). Many different methods have been used to approximate the solution of Hammerstein integral equations (Kumar, S., I.H. Sloan, 1987; Wang, X., W. Lin, 1998; Kumar, S., I.H. Sloan, 1987; Abou El-seoud, A.A.M., 2003).

The double exponential transformation was first proposed by Takahasi and Mori (1974). It has become widely used for numerical solution of problems in applied physics and engineering. Optimality of the double exponential formula-functional analysis approach has been proposed by Sugihara (1997). A review of the double-exponential transformation in numerical integration and in a variety of Sinc numerical methods has been presented in (Mori, M., M. Sugihara, 2001). In (Sugihara, M., 2002; Sugihara, M., T. Matsuo, 2004; Nurmuhammada, A., 2005; Wu, X., W. Kong, C. Li, 2006; Nurmuhammada, A., 2007) the double exponential transformation methods are developed for solving the boundary value problems. Numerical solution of indefinite
integrals and linear integral equations by means of the Sinc method based on the double exponential transformation has also been employed in (Muhammad, M., M. Mori, 2003; Muhammad, M., 2005).

The Sinc method for the nonlinear Hammerstein integral equations has been considered in (Rashidinia, J., M. Zarebnia, 2007) based on the single exponential transformation. In this paper a global approximation for the solution of the Eq. (2) and (3) using the double exponential transformation is developed.

This work is organized as follows. First, in section 2 we review some of the main properties of Sinc function and double exponential(DE) transformation that are necessary for the formulation of the discrete systems. In section 3, we illustrate how the DE method may be used to replace equations (2) and (3) by an explicit systems of nonlinear algebraic equations, which are solved by Newton’s method. In section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

2 Preliminaries:

The Sinc function is dened on the real line by

\[
Sinc(x) = \begin{cases} 
\sin(\pi x), & x \neq 0; \\
1, & x = 0.
\end{cases}
\]  

For any h > 0, the translated Sinc functions with evenly spaced nodes are given as

\[
S(j, h)(x) = Sinc\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \cdots,
\]  

which are called the jth Sinc function (Stenger, F., 1993). The Sinc function form for the interpolating point is given by

\[
S(j, h)(kh) = \delta^{(0)}_{jk} = \begin{cases} 
1, & k = j; \\
0, & k \neq j.
\end{cases}
\]  

Let D be a simply-connected domain, having boundary \(\partial D\). Let a and b denote two distinct points of \(\partial D\) and denote a conformal map of \(D_a\) onto D, where \(D_a\) denote the region \(\{w \in \mathbb{C} : |\text{Im}t| < d\} \) such that \(\varphi(\infty) = a\) and \(\varphi(\infty) = b\). Let \(\varphi^{-1}(D_a)\) denote the inverse map. Given \(\varphi\) and a positive number h, let us set \(x_k = \varphi(kh), k = 0, \pm 1, \pm 2, \ldots\).

Let \(H'(D_a)\) the family of all functions f analytic in \(D_a\), such that if \(D_a(\varepsilon)\) is defined for \(0 < \varepsilon < 1\) by

\[
D_a(\varepsilon) = \{t \in \mathbb{C} : |\text{Re}t| < 1/\varepsilon, |\text{Im}t| < d(1 - \varepsilon)\},
\]

then \(N_1(f; D_a) < \infty\), where

\[
N_1(f, D_a) = \lim_{\varepsilon \to 0} \int_{\partial D_a(\varepsilon)} |f(t)||dt|.
\]

A function f is said to decay double exponentially with respect to the conformal map \(\varphi\) if there exist positive constants \(\infty\) and C such that

\[
|f(\varphi(t))\varphi'(t)| \leq C\exp(-\alpha \exp|t|), \quad t \in (-\infty, \infty).
\]  

Now, we introduce the space \(K^a_\varphi(D_a)\) (Muhammed. M et al, 2005). Let be a positive number, then \(K^a_\varphi(D_a)\) denotes the family of function \(f\) where \(f(\varphi(t))\varphi'(t)\) belongs to \(H'(D_a)\) and the analytic
function \( f \) satisfies the equation (7).

To construct approximation on the interval \([a, b]\), we consider the conformal map

\[
x = \phi(t) = \frac{(b - a)}{2} \tanh\left(\frac{\pi}{2} \sinh t\right) + \left(\frac{b + a}{2}\right),
\]

\[
\phi'(t) = \frac{(b - a)}{2} \left(\frac{\pi}{2} \cosh t\right) \cosh\left(\frac{\pi}{2} \sinh t\right).
\]  

(8)

Let \( u \in K^a_\phi(D_d) \), let \( N \) be a positive integer, and let mesh size \( h \) be selected by the formula

\[
h = \frac{1}{N} \log\left(\frac{2\pi dN}{\alpha}\right),
\]

then, we have the double exponential formula for the definite integration of a function \( u \) as (Takahasir, H., 1974):

\[
\int_1^u u(x)dx = h \sum_{k=-N}^N u(\phi(k))\phi'(kh) + O\left(\exp\left(-\frac{2\pi dN}{\log(2\pi dN/\alpha)}\right)\right).
\]  

(10)

Also, let \( u \in K^a_\phi(D_d) \), and let \( h = \frac{1}{N} \log(\pi dN/\alpha) \). Then we have the indefinite integration formula based on the double exponential transformation for the function \( u \) as follows (Takahasir, H., 1974):

\[
\int_a^\infty u(x)dx = h \sum_{k=-N}^N u(\phi(kh))\phi'(kh) \left(\frac{1}{2} + \frac{1}{\pi} Si\left(\pi \phi^{-1}(s) - kn\right)\right)
+ O\left(\frac{\log N}{N} \exp\left(-\frac{\pi dN}{\log(\pi dN/\alpha)}\right)\right),
\]

(11)

where

\[
Si(x) = \int_0^x \frac{\sin t}{t} dt.
\]

3 The Approximate Solution of Hammerstein Integral Equations:

3.1 Volterra- Hammerstein integral equation. we consider the nonlinear Volterra-Hammerstein integral equation of the form

\[
u(x) = f(x) + (KHu)(x), \quad x \in [a, b],
\]  

(12)

where

\[
(KHu)(x) = \int_a^x K(x, t)H(t, u(t))dt,
\]

(13)

In Eq. (12) \( f \) and the kernel \( K \) are continuous functions, and also \( H(t, u) \) is nonlinear in \( u \). For convenience, consider
For the right-hand side of (13), we suppose that $K(x, \cdot)Q(\cdot) \in K^\nu_\phi(D_j)$, then by using the indefinite integration formula (11), we obtain:

$$
\int_0^x K(x,t)Q(t)\,dt \approx h \sum_{j=-N}^N K(x, t_j)Q_j \phi'(jh) \left( \frac{1}{2} + \frac{1}{\pi} \text{Si} \left( \frac{\phi^{-1}(x)}{h} - j\pi \right) \right), \tag{14}
$$

where $\phi'$ is defined by (8), and also,

$$
Q_j = Q(t_j) = H(t_j, u(t_j)), \quad x = \phi(\tau), \quad -\infty < \tau \leq \phi^{-1}(x),
$$

$$
t_j = \phi(jh), \quad j = -N, \ldots, N. \tag{15}
$$

Having replaced the second term on the right-hand side of (12) with the right hand side of (14), and having substituted $x = x_k$ for $k = -N, \ldots, N$, that $x_k$ are Sinc grid points as,

$$
x_k = \phi(kh) = \frac{(b-a)}{2} \text{tanh} \left( \frac{\pi}{2} \sinh kh \right) + \frac{(b+a)}{2}, \quad k = -N, \ldots, N, \tag{15}
$$

we get the collocation result:

$$
u_k - h \sum_{j=-N}^N K(x_k, t_j)Q_j \phi'(jh) \left( \frac{1}{2} + \frac{1}{\pi} \text{Si} \left( \frac{\phi^{-1}(x_k)}{h} - j\pi \right) \right) = f(x_k), \quad k = -N, \ldots, N, \tag{16}
$$

where $u_k$ denotes an approximate value of $u(x_k)$. The system in (16) can be simplified in the matrix form as:

$$
U - (K \ast Y)Q = F. \tag{17}
$$

The notation “$\ast$” denotes the Hadamard matrix multiplication. We denote $K = [K(x_k, t_j) \phi'(jh)]$, $k, j = -N, \ldots, N$ and $Y = [\Omega_{h,j}(x_k)]$, $k, j = -N, \ldots, N$, which are square matrices of order $(2N + 1)\times(2N + 1)$, and also:

$$
U = [u_{-N}, u_{-N+1}, \ldots, u_N]^T, \quad \Omega_{h,j}(x_k) = \left( \frac{1}{2} + \frac{1}{\pi} \text{Si} \left( \frac{\phi^{-1}(x_k)}{h} - j\pi \right) \right),
$$

$$
Q = [Q_{-N}, Q_{-N+1}, \ldots, Q_N]^T, \quad F = [f_{-N}, f_{-N+1}, \ldots, f_N]^T.
$$

The above nonlinear system consists of $2N+1$ equations with $2N+1$ unknowns \( \{u_j\}_{j=-N}^N \). Solving this nonlinear system by Newton’s method, we obtain an approximate solution $u_j, j = -N, -N + 1, \ldots, N$ with correspond to the exact solution $u(x_j), j = -N, -N + 1, \ldots, N$ of the Volterra-Hammerstein integral equations (12) at the Sinc points. Having used the approximate solution $u_j, j = -N, -N + 1, \ldots, N$, we employ a method similar to the Nystroms idea for the Volterra-Hammerstein integral equation (Stenger, F., 1993), i.e., we use

$$
u_N(x) = f(x) + h \sum_{j=-N}^N K(x, t_j)H(t_j, u_j)\phi'(jh)\Omega_{h,j}(x), \tag{18}
$$
where

\[ \Omega_{h,j}(x) = \left( \frac{1}{2} + \frac{1}{\pi} \sin\left( \frac{\phi^{-1}(x) j \pi}{h} \right) \right). \]

3.2 Fredholm- Hammerstein integral equation. Consider the Hammerstein integral equation (3) and let

\[ P(t) = H(t, u(t)) \]

then we have

\[ u(x) = f(x) + \int_a^b K(x, t) P(t) dt, \quad x \in [a, b]. \] (20)

Let \( K(x, \cdot) P(\cdot) \in K^a \phi (D_a) \) then by applying the formula (10), we get

\[ \int_a^b K(x, t) P(t) dt \approx h \sum_{j=-N}^{N} K(x, t_j) P_j \phi'(j h), \] (21)

where

\[ h = \frac{1}{N} \log \left( \frac{2\pi d N}{\alpha} \right), \quad t_j = \phi(j h), \quad j = -N, \ldots, N; \]
\[ P_j = P(t_j) = H(t_j, u(t_j)), \quad x = \phi(\tau), \quad -\infty < \tau \leq \phi^{-1}(x). \]

By replacing the second term on the right-hand side of (20) with right-hand side of (21), and substituting \( x = x_k \), that is defined by (15), we obtain

\[ u_k - h \sum_{j=-N}^{N} K(x_k, t_j) P_j \phi'(j h) = f(x_k), \quad t_j = \phi(j h), \quad k = -N, \ldots, N. \] (22)

There are \((2N + 1)\) unknowns \( u_j, j = -N, -N + 1, \ldots, N \). In order to determine these \((2N + 1)\) unknowns, we rewrite this system which is the nonlinear system of equations in matrix form as:

\[ U - KP = 0; \] (23)

where

\[ U = [u_{-N}, u_{-N+1}, \ldots, u_N]^T, \]
\[ K = [K(x_k, t_j) \phi'(j h)], \quad k, j = -N, \ldots, N; \]
\[ P = [P_{-N}, P_{-N+1}, \ldots, P_N]^T, \]
\[ \Theta = [f_{-N}, f_{-N+1}, \ldots, f_N]^T. \]

Solving the nonlinear system (23) by Newton’s method, we obtain an approximate solution \( u_j, j = -N, -N + 1, \ldots, N \). Then, we can obtain an approximation to the solution (20) as follows:

\[ u_N(x) = f(x) + h \sum_{j=-N}^{N} K(x, t_j) P(t_j, u_j) \phi'(j h). \] (24)
4 Numerical Examples:

In order to illustrate the performance of the Sinc method based on the double exponential transformation in solving Hammerstein integral equations and justify the accuracy and efficiency of the presented method, we consider the following examples. Also, these examples have been solved by the single exponential method given in (Rashidinia, J., 2007). The errors are reported for the single exponential(SE) and double exponential(DE) methods. The maximum of absolute error for the SE method on the set of grid points

$$S_{SE} = \{x_{-N}, \ldots, x_0, \ldots, x_N\},$$

$$x_k = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = -N, \ldots, N,$$  \hspace{1cm} (25)

is

$$\|E_{SE}\|_{\infty} = \max_{-N \leq j \leq N} |u(x_j) - u_N(x_j)|,$$  \hspace{1cm} (26)

and also for the DE method, maximum of absolute error is given as:

$$\|E_{DE}\|_{\infty} = \max_{-N \leq j \leq N} |u(x_j) - u_N(x_j)|,$$  \hspace{1cm} (27)

where

$$S_{DE} = \{x_{-N}, \ldots, x_0, \ldots, x_N\},$$

$$x_k = \phi(kh), \quad k = -N, \ldots, N.$$  \hspace{1cm} (28)

We stopped the numbers of iteration in the Newton’s method when, the distance between two iteration is less than a given tolerance, $$\varepsilon = 10^{-4},$$

$$\|u^{(i+1)} - u^{(i)}\| \leq \varepsilon$$

The numerical results are tabulated in Tables 1, 2 and shown in Figures 1, 2.

Example 1:

Consider the following Volterra-Hammerstein integral equation with exact solution $$u(x) = e^x,$$

$$u(x) = e^x - (x + 1)\sin x + \int_{-1}^{x} e^{-2t} \sin xu^2(t)dt, \quad -1 \leq x \leq 1.$$  \hspace{1cm} (29)

We solve (29) for different values of $$N$$ by using SE and DE methods. The maximum absolute errors for the DE and SE transformations are tabulated in Table 1. The optimal mesh size for the DE transformation is given by $$h = \frac{1}{N} \log(\frac{\pi N}{2})$$ since $$d = \frac{\pi}{4}, \quad a = \frac{\pi}{2}.$$ This Table indicate that as $$N$$ increases the errors decrease more rapidly. It should be mentioned that the errors on the grid points $$S_{DE}$$ decrease more rapidly than the errors on the grid points $$S_{SE}.$$ The exact and approximate solutions of example 1 are plotted in Fig. 1 for $$N = 1$$ and $$N = 2.$$ For large values of $$N$$ the approximate solution is indistinguishable from the exact solution.
Table 1: Results for Example 1.

| \( N \) | \( |E_{|g21}^{2}| \) | \( |E_{|g21/g23}^{2}| \) |
|---|---|---|
| 2 | 1.85334 ×10^{-3} | 2.23814 ×10^{-2} |
| 4 | 4.12882 ×10^{-4} | 1.04225 ×10^{-2} |
| 6 | 3.16428 ×10^{-5} | 4.54005 ×10^{-3} |
| 8 | 2.87138 ×10^{-6} | 2.12286 ×10^{-3} |
| 10 | 3.00121 ×10^{-7} | 1.06267 ×10^{-3} |

Fig. 1: Exact and approximate solutions for Example 1, \((N = 1, N = 2)\).

Example 2: Consider the Fredholm-Hammerstein integral equation

\[
\int_0^1 \left( x + t \right) e^{x(t)} \, dt, \quad 0 \leq x \leq 1,
\]

with exact solution \( u(x) = x \).

The approximate solutions are calculated by SE and DE methods for different values of \( N \) and the optimal mesh size \( h = \frac{1}{N} \log(\frac{N}{2}) \). The results are tabulated in Table 2 and shown in Figure 2. We can observe similar results to Example 1.

Fig. 2: Exact and approximate solutions for Example 2, \((N = 1, N = 2)\).
Table 2: Results for Example 2.

<table>
<thead>
<tr>
<th>N</th>
<th>$|E_{DE}|_{\infty}$</th>
<th>$|E_{SE}|_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2.42311 \times 10^{-3}$</td>
<td>$9.94991 \times 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>$2.05663 \times 10^{-5}$</td>
<td>$1.64235 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>$1.34363 \times 10^{-7}$</td>
<td>$4.63475 \times 10^{-4}$</td>
</tr>
<tr>
<td>8</td>
<td>$1.67297 \times 10^{-8}$</td>
<td>$1.53360 \times 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>$7.37791 \times 10^{-10}$</td>
<td>$5.70642 \times 10^{-5}$</td>
</tr>
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</table>

**Conclusion:**

In this paper the SE and DE methods applied to find the approximate solutions of Hammerstein integral equations of Volterra and Fredholm types. The numerical solutions are given in tables (1), (2) and figures (1), (2). The results showed that the double exponential transformation method is more powerful than single exponential method. However, in this paper we showed application of the double exponential transformation for solving Hammerstein integral equations.

**REFERENCES**


