Australian Journal of Basic and Applied Sciences, 4(8): 3506-3512, 2010
ISSN 1991-8178

Super-efficiency in Data Envelopment Analysis: An Application to Gas Companies

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Abstract: Conventional data envelopment analysis (DEA) assists decision makers in distinguishing between efficient and inefficient decision making units (DMUs) in a homogeneous group. However, DEA does not provide more information about efficient units. Super-efficiency DEA models can be used in ranking the performance of efficient DMUs. This research proposes a methodology to determine an Euclidean distance-based measure of super-efficiency. Then, the DMUs are ranked according to their super-efficiency score. The applicability of the proposed model is illustrated in the context of the analysis of gas companies performance.

Key words: Data envelopment analysis, ranking, super-efficiency.

INTRODUCTION

Data envelopment analysis(DEA) is a data-oriented method for measuring and benchmarking the relative efficiency of peer decision making units (DMUs) with multiple inputs and multiple outputs. DEA was initiated in 1978 when Charnes, Cooper and Rhodes (CCR) demonstrated how to change a fractional linear measure of efficiency into a linear programming format. This non-parametric approach solves an LP formulation per DMU and the weights assigned to each linear program are the results of the corresponding LP. The weights are chosen so as to show the specific DMU in as positive a light as possible under the restrictions that no other DMU given the same weights, is more than %100 efficient. In DEA, the best performers have the full-efficient status denoted by unity, and from experience we know that plural DMUs have this efficient status. To discriminate between these efficient DMUs is an interesting subject. Super-efficiency DEA model is obtained when a DMU under evaluation is excluded from the reference set of the original DEA model. Many attempts have been made to determine a scalar measure of super efficiency in DEA. See, for instances, Tone (2002), Bogetoft (2004), Bernroider (2007), Chen (2004, 2005), Liu and Peng (2008), Shanling (2007) and etc. Because the possible infeasibility of super-efficiency DEA models, the use of these models has been restricted to the situations where constant returns to scale are assumed.

In this paper, we will propose a super efficiency measure based on the Euclidean distance. Our Euclidean distance based (EDB) measure of super-efficiency can be used to fully ranking of efficient units. If our EDB measure of super-efficiency is applied in ranking of efficient units, infeasibility does not occur. We consider the super-efficiency issues under the constant returns to scale environment and the results can be extended to the variable returns to scale case. An illustrative application of the methodology to a sample of gas companies from twenty five regions of Iran is given. The rest of this paper is organized as follows: the next section of the paper presents the various DEA models. Then, we introduce our EDB-measure of super efficiency. Section four applies the method to an example involving 25 Iranian gas companies. Conclusions appear in section five.

2 Preliminaries:

Suppose we have n DMUs, $DMU_j : j = 1, \cdots, n$, which produce s outputs, $y_{rj} : r = 1, \cdots, s$ by utilizing m inputs, $x_{ij} : i = 1, \cdots, m$. Relative efficiency is defined as the ratio of total weighted outputs to the total weighted inputs. The efficiency measure for DMU_o is defined as

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$$e_o = \sum_{r=1}^{s} u_r y_{ro} \Big/ \sum_{i=1}^{m} v_i x_{io}$$

where the weights u_r and v_i are non-negative. To estimate the DEA efficiency of DMU_o , we use the following original DEA model of Charnes, Cooper and Rhodes(1978):

$$\begin{aligned} Max \quad & e_o = \sum_{r=1}^{s} u_r y_{ro} \Big/ \sum_{i=1}^{m} v_i x_{io} \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \le 1, \quad j = 1,\dots,n, \\ & u_r \, , v_i \ge \varepsilon, \quad \text{for all i, } r, \end{aligned} \tag{1}$$

where ε >0 is a non-archimedean construct. The efficiency ratio in (1) ranges between zero and one, with DMU_o being considered relatively efficient if it receives a score of one. The above formulation can be translated into a linear program using the method of Charnes and Cooper (1962) as

$$\begin{aligned} Max \quad & e_o = \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n, \\ & u_r \, , v_i \ge \varepsilon, \quad \text{for all i, } r. \end{aligned} \tag{2}$$

All the frontier DMUs (efficient DMUs) have $e_o = 1$. In order to discriminate the performance of efficient DMUs, Andersen and Petersen (1993) developed a procedure for ranking efficient units. Their methodology enables an extreme efficient unit *o* to achieve an efficiency score greater than one by removing the *o-th* constraint in (2) as shown in model (3):

$$\begin{aligned} Max \quad & \phi_o = \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,...,n,\ j \ne o, \\ & u_r\ , v_i \ge \varepsilon, \quad \text{for all i, } r. \end{aligned} \tag{3}$$

Let the optimal objective value to (3) be ϕ_o . For an efficient DMU_o , ϕ_o is not less than unity and this value indicates *super-efficiency* of DMU_o . However, in many real applications, (3) may be infeasible and unstable. Tone (2002) has defined the super SBM efficiency of DMU_o as the optimal objective function value δ_o of the following program:

$$\begin{aligned} Min \quad & \delta_o = \frac{\frac{1}{m}\sum_{i=1}^{m} \frac{\overline{x}_i}{x_{io}}}{\frac{1}{s}\sum_{r=1}^{s} \frac{\overline{y}_r}{y_{ro}}} \\ s.t. \quad & \sum_{\substack{j=1 \\ j\ne o}}^{n} \lambda_j x_{ij} \le \overline{x}_i,\ i = 1,\cdots,m, \\ & \sum_{\substack{j=1 \\ j\ne o}}^{n} \lambda_j y_{rj} \ge \overline{y}_r,\ r = 1,\cdots,s, \\ & \overline{x}_i \ge x_{io}, \quad i = 1,\cdots,m, \\ & 0 \le \overline{y}_r \le y_{ro}, \quad r = 1,\cdots,s, \\ & \lambda_j \ge 0, \quad \text{j} = 1,\cdots,\text{n}. \end{aligned} \tag{4}$$

δ_o is a weighted L_1 distance from (x_o, y_o) to the production possibility set spanned by $(x_j, y_j),\ j = 1,\cdots,n,\ j \ne o.$

3 A Measure of Super Efficiency:

In this section, we discuss the super-efficiency issue under the assumption that the DMU_o is CCR-efficient. The production possibility set $T_{(x_o,y_o)}$ spanned by $(x_j, y_j): j = 1, 2,...,n,\ j \ne o$ is defined as

$$T_{(x_o,y_o)} = \{(x,y): x \geq \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_J x_j\ ,\ \ y \leq \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_J y_j,\ \ \lambda_j \geq 0,\ \ j=1,\dots,n,\ \ j \neq o\} \tag{5}$$

Obviously, $T_{(x_o,y_o)}$ is not empty. Before introducing our super efficiency measure, we provide some definitions.

Definition 1- A surface $H = \{(x,y): -\alpha^t x + \beta^t y = 0, \alpha \geq 0\,; \beta \geq 0\} \bigcap T_{(x_o,y_o)}$ is called an efficient supporting surface of $T_{(x_o,y_o)}$ if for each extreme efficient observation *j*, $-\alpha^t x_j + \beta^t y_j \leq 0.$

Definition 2- Let $H = \{(x,y): -\alpha^t x + \beta^t y = 0\}$ is a hyperplane. If $(\hat{x},\hat{y}) \notin H$, the point in *H* nearest to $(\hat{x},\hat{y})$ in terms of the Euclidean distance is

$$(\overline{x},\overline{y}) = (\hat{x},\hat{y}) + \frac{-\alpha^t \hat{x} + \beta^t \hat{y}}{(\|(\alpha,\beta)\|_2)^2}(-\alpha,\beta).$$

It can be verified that the straight line joining $(\overline{x},\overline{y})$ and $(\hat{x},\hat{y})$ is perpendicular to the hyperplane at $(\hat{x},\hat{y})$.

Definition 3- (Reference supporting surface) For a DMU_o a reference supporting surface is an efficient surface of $T_{(x_o,y_o)}$ that contains the reference units of DMU_o .

Definition 1 states that an efficient supporting surface of $T_{(x_o,y_o)}$ has to support at least one extreme efficient unit. The vector α and β are selected such that at least one inequalities of $-\alpha^t x_j + \beta^t y_j \leq 0$ is binding. Toward this end, we use the slack variables s_j and rewrite the inequalities as $-\alpha^t x_j + \beta^t y_j + s_j = 0.$ Let *E* is the set of indices corresponding to all extreme efficient units. Constraints $s_j \leq (1-\gamma_j)M$, $\gamma_j \in \{0,1\},$ for $, j \in E,$ and $\sum_{j\in E} \gamma_j \geq 1,$ force at least one of the s_j at zero level (*M* is a large positive constant). Clearly, selecting $\gamma_t = 1$ forces the $s_t = 0$. Hence, to determine a reference supporting surface of DMU_o , we solve the following mixed-integer linear program

$$\begin{aligned}
Max \quad & \rho_o = -\alpha^t x_o + \beta^t y_o \\
s.t. \quad & -\alpha^t x_j + \beta^t y_j + s_j = 0,\ \ j \in E,\ \ j \neq o, \\
& 1\alpha + 1\beta = 1, \\
& s_j \leq (1-\gamma_j)M, j \in E, \\
& \sum_{j\in E} \gamma_j \geq 1, \\
& \alpha \geq 0, \beta \geq 0, \gamma_j \in \{0,1\},\ \ s_j \geq 0,\ \ j \in E.
\end{aligned} \tag{6}$$

in which $1=(1,1,\cdots,1)$. It is to be noted that in the first constraints of (6) we exclude inefficient units, this means that DMU_o will be evaluated with efficient units. Let an optimal solution to (4) be $(\bar{\alpha},\bar{\beta})$. Obviously, $\hat{H}=\{(x,y):-\bar{\alpha}^t x+\bar{\beta}^t y=0\}\bigcap T_{(x_o,y_o)}$ is an efficient surface of $T_{(x_o,y_o)}$. By the definition 2, the orthogonal projection of (x_o,y_o) in $\hat{H}$ is $(\hat{x},\hat{y})=(x_o,y_o)+\mu(\bar{\alpha},-\bar{\beta})$ in which $\mu=\dfrac{-\bar{\alpha}^t x_o+\bar{\beta}^t y_o}{\left(\left\|(\bar{\alpha},\bar{\beta})\right\|_2\right)^2}$. Clearly, $\mu\ge 0$, because, DMU_o is CCR-efficient and $(x_o,y_o)\notin T_{(x_o,y_o)}$.

Let $E_o=\{i:x_{io}>0\}$ and $F_o=\{r:y_{ro}>0\}$. We define the EDB-super-efficiency index $\pi_o(x_o,y_o)$ as

$$\pi_o(x_o,y_o)=\left(\frac{1}{Card(E_o)}\sum_{i\in E_o}\frac{x_{io}+\mu\bar{\alpha}_i}{x_{io}}\right)\Bigg/\left(\frac{1}{Card(F_o)}\sum_{r\in F_o}\frac{y_{ro}-\mu\bar{\beta}_r}{y_{ro}}\right) \qquad (7)$$

In which $Card(.)$ is the cardinality of a set.

We illustrate the EDB-super efficiency measure with a small-scale example consisting of five *DMU*s. The *DMU*s use two inputs to produce a single output whose value is normalized to one for each *DMU*. Note that DMU_5 has zero element in the first input, e.g. $x_{15}=0$. This means that DMU_5 has no function as to the input 1. Also, DMU_1 has a small element in the second input. The CCR model (1) indicates that all *DMU*s are efficient. It can be seen that model (3) is infeasible when DMU_5 is under evaluation and this model yields to a large score 11.111 to DMU_1. We have calculated the super EDB measure for each *DMU*. The data set, the CCR-efficiency score e_o, the super efficiency score ϕ_o and the super EDB measure π_o are listed in table 1. Our approach shows that DMU_1 is the top-ranked *DMU* followed by DMU_5, DMU_2, DMU_3 and DMU_4.

Table 1: The data and results used in the simple example

DMU_j	x_1	x_2	y_1	e_o	ϕ_o	π_o
1#	17	0.09	1	1.00	11.111	(1) 3.987
2#	2	4	1	1.00	1.333	(3) 1.257
3#	6	2	1	1.00	1.167	(4)1.129
4#	11	1	1	1.00	1.045	(5)1.036
5#	0	8	1	1.00	infeasible	(2)2.44

The number in parentheses represents rank.

4 An Empirical Study:

This section illustrates the EDB-super efficiency measure discussed in this paper with the analysis of gas companies' activities. The data set consists of 25 gas companies located in 24 regions in Iran. The data for this analysis are derived from operations during 2005. We have used seven variables from the data set as inputs and outputs. Inputs include capital (x_1), number of staff (x_2) and operational costs(excluding staff costs) (x_3), and outputs include number of subscribers (y_1), amount of pipe-laying (y_2), length of gas network (y_3) and the revenue of sold-out gas (y_4). The chosen input-output normalized data that are used in the application are displayed in table 2.

Table 2: Iranian gas companies

#j	x_1	x_2	x_3	y_1	y_2	y_3	y_4
1	0.4296	0.3307	0.3020	0.7207	1.0000	0.2425	0.5974
2	0.2380	0.1419	0.2315	0.5773	0.4308	0.1425	0.4408
3	0.1210	0.1196	0.0983	0.1665	0.1880	0.0861	0.1258
4	0.4939	0.8860	0.8874	0.5409	0.8297	0.8615	0.5050
5	1.0000	1.0000	1.0000	0.4280	0.1924	1.0000	0.2425
6	0.2531	0.3542	0.6211	1.0000	0.7201	0.4242	1.0000
7	0.0742	0.1285	0.1543	0.0927	0.2679	0.0655	0.0986
8	0.7381	0.5084	0.6143	0.6304	0.7616	0.5534	0.4890
9	0.3936	0.5654	0.4209	0.2363	0.2711	0.3987	0.3372
10	0.0978	0.0983	0.1479	0.0632	0.0995	0.0654	0.0546
11	0.0680	0.1296	0.1216	0.0856	0.1717	0.0840	0.0511
12	0.4236	0.6458	0.5849	0.4745	0.7316	0.3232	0.3539
13	0.1110	0.1151	0.1614	0.1397	0.2486	0.1667	0.1408
14	0.0805	0.1151	0.1130	0.0927	0.0749	0.0864	0.0667
15	0.1193	0.0905	0.1046	0.1865	0.1826	0.0984	0.1775
16	0.3805	0.1698	0.2739	0.2453	0.5380	0.1925	0.1081
17	0.2408	0.1073	0.0727	0.2679	0.2517	0.1178	0.1855
18	0.0438	0.1162	0.0662	0.0374	0.0892	0.0122	0.0280
19	0.1133	0.2804	0.1381	0.1809	0.2743	0.0908	0.1851
20	0.2544	0.4335	0.2795	0.2794	0.5248	0.2344	0.2206
21	0.1147	0.1207	0.0931	0.2528	0.2100	0.0956	0.2107
22	0.2961	0.4201	0.2935	0.7029	0.9879	0.3020	0.7303
23	0.0998	0.1777	0.1940	0.1896	0.2760	0.2210	0.1786
24	0.2340	0.3687	0.1541	0.1876	0.4625	0.1431	0.1679
25	0.1333	0.1196	0.1426	0.2970	0.4014	0.1305	0.3002

In table 3 we have recorded different efficiency measures. The *2-th* and *3-th* columns of the table report the optimal values to models (2) and (3), respectively. The CCR model indicates that ten companies (#1, #2, #6, #7, #13, #17, #21, #22, #23 and #25) are full-efficient(see column 2 in table 3). The fourth column of the table reports the super-EDB measure of efficiency π_0 defined in (7). As the table indicates, company #6 is the top-ranked company followed by #2, #17, #23, #25, #22, #13, #1, #7 and #21. We have also calculated the slack-based super-efficiency measure of Tone for each company. The last column of the table reports this measure. (The number in parentheses represents rank.)

5 Conclusions:

In this paper, we have proposed a super-efficiency measure based on the Euclidean distance. Our Euclidean distance based (EDB) measure of super-efficiency can be used to fully ranking of efficient units. The rationality for this measure is to determine the Euclidean distance from the unit under evaluation to efficient frontier. If our EDB measure of super-efficiency is applied in ranking of efficient units, infeasibility does not occur. The applicability of the models developed is illustrated in the context of the analysis of gas companies performance. We consider the super-efficiency issues under the constant returns to scale environment. We can easily extend our analysis to the variable returns to scale case.

Table 3: The result for the empirical study

$j\#$	e_o	ϕ_o	π_o	δ_o
1	1	(9)1.0803	(8)1.0262	(9)1.0190
2	1	(3)1.4605	(2)1.1249	(3)1.1284
3	0.7687	-	-	-
4	0.8461	-	-	-
5	0.8202	-	-	-
6	1	(1) 1.6620	(1)1.1792	(1)1.2888
7	1	(8)1.0818	(9)1.0202	(8)1.0193
8	0.8033	-	-	-
9	0.7812	-	-	-
10	0.4675	-	-	-
11	0.8008	-	-	-
12	0.5475	-	-	-
13	1	(6)1.1827	(7)1.0391	(7)1.0453
14	0.6602	-	-	-
15	0.9300	-	-	-
16	0.9929	-	-	-
17	1	(2)1.5481	(3)1.1068	(2)1.2081
18	0.5873	-	-	-
19	0.7281	-	-	-
20	0.7543	-	-	-
21	1	(10)1.0665	(10)1.0199	(10)1.0187
22	1	(7)1.1798	(6)1.0489	(5)1.0896
23	1	(4)1.3110	(4)1.0910	(6)1.0850
24	0.8856	-	-	-
25	1	(5)1.2381	(5)1.0748	(4)1.1212

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