Australian Journal of Basic and Applied Sciences, 4(8): 3513-3518, 2010
ISSN 1991-8178

Quadratic Programming with Fuzzy Numbers: A Linear Ranking Method

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Abstract: Quadratic programming has been widely applied to solve real world problems. The conventional quadratic programming model requires the parameters to be known constants. In the real world, however, the parameters are seldom known exactly and have to be estimated. In this paper, we define a quadratic programming with fuzzy numbers (QPFN) problem where the cost coefficients, constraint coefficients, and right-hand sides are represented by trapezoidal and/or triangular fuzzy numbers. Moreover, we give some important results concerning to the conditions of the existence of a unique solution. Then we will discuss how can solve these problems by using linear ranking functions.

Key words: Fuzzy numbers, fuzzy quadratic programming, ranking function.

INTRODUCTION

Quadratic programming is a mathematical modeling technique designed to optimize the usage of limited resources. It has led to a number of interesting applications and the development of numerous useful results (Abdel-Malck and Areeractch, 2007; Ammar and Khalifa, 2003; Dwyer *et al.,* 2006; Pavlovic and Divnic, 2007; Schwarz, 2006). A quadratic programming problem is an optimization problem involving a quadratic objective function and linear constraints. Liu describes a solution method for solving a class of fuzzy quadratic programming problems, where the cost coefficients of the linear terms in objective function, constraint coefficients, and right-hand sides are fuzzy numbers. After that he generalized his first method to a more general fuzzy quadratic programming problem, where the cost coefficients in objective function, constraint coefficients, and right-hand sides are all fuzzy numbers. A pair of twolevel mathematical programs is formulated to calculate the upper bound and lower bound of the objective values of the fuzzy quadratic program. Based on the duality theorem and by applying the variable transformation technique, the pair of two-level mathematical programs is transformed into a family of conventional one-level quadratic programs. Solving the pair of quadratic programs produces the fuzzy objective values of the problem. In this paper, we consider a fuzzy version of the quadratic programming problems and then present a method to solve these problems by using linear ranking functions as a convenient tools for ordering fuzzy numbers (see (Mahdavi-Amiri and Nasseri, 2007; 2006)).

The paper is organized in 5 Sections. In the next Section, we give some necessary notations and definitions of fuzzy set theory and fuzzy arithmetic. Section 3 provides a discussion of fuzzy numbers and linear ranking functions for ordering them. In particular, Yager's linear ranking function for ordering trapezoidal fuzzy numbers is emphasized. We define the quadratic programming with fuzzy numbers (QPFN) problems in Section 4, and focus on solving theses problems. Moreover, we give a key theorem to state some conditions of the existence of a unique solution. Finally We conclude in Section 5.

Definitions and Notations:

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh (1970).

Definition 2.1:

Let X be the universal set. $\tilde{A}$ is called a fuzzy set in X if $\tilde{A}$ is *a* set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

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where $\mu_{\tilde{A}}(x)$ is the membership function of $\tilde{A}$.

Definition 2.2:
The λ-level set of $\tilde{A}$ is the set

$$\tilde{A}_\lambda = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \lambda\}$$

where λ ∈ (0, 1]. The lower and upper bounds of any λ-level set $\tilde{A}_\lambda$ are represented by finite numbers

$\inf_{x \in \tilde{A}_\lambda}$ and $\sup_{x \in \tilde{A}_\lambda}$.

Definition 2.3:
A convex fuzzy set $\tilde{A}\, on$□ is a fuzzy number if the following conditions hold:

(a)Its membership function is piecewise continuous.
(b)There exist three intervals [*a, b*], [*b, c*] and [*c, d*] such that $\mu_{\tilde{A}}$ is increasing on [*a, b*], equal to 1 on [*b, c*], decreasing on [*c, d*] and equal to 0 elsewhere.

Definition 2.4:

Let $\tilde{A} = (a^L, a^U, \alpha, \beta)$ denote the trapezoidal fuzzy number, where $(a^L - \alpha, a^U + \beta)$ is the support of $\tilde{A}$ and $[a^L, a^U]$ its core.

Remark 2.1:
We denote the set of all trapezoidal fuzzy numbers by F(□) .

We next define arithmetic on trapezoidal fuzzy numbers. Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers. Define,

$$x \geq 0, x \in \mathbb{R}; \quad x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta)$$
$$x < 0, x \in \mathbb{R}; \quad x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha)$$
$$\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta)$$

We point out that the arithmetic on trapezoidal fuzzy numbers follow the Extension Principle (see (Lai and Hwang, 1992)).

Ranking Functions:
An effective approach for ordering the elements of $F(\mathbb{R})$ is to define a ranking function $R : F(\mathbb{R}) \to \mathbb{R}$ which maps each fuzzy number into the real line, where a natural order exists.
We define orders on $F(\mathbb{R})$ by:

$\tilde{a} \succeq \tilde{b}$ If and only if $R(\tilde{a}) \geq R(\tilde{b})$ (3.1)

$\tilde{a} \succ \tilde{b}$ If and only if $R(\tilde{a}) > R(\tilde{b})$ (3.2)

$\tilde{a}$ □ $\tilde{b}$ If and only if $R(\tilde{a}) = R(\tilde{b})$ (3.3)

where $\tilde{a}$ and $\tilde{b}$ are in $F(\mathbb{R})$. Also we write $\tilde{a} \succeq \tilde{b}$ if and only if $\tilde{b} \succeq \tilde{a}$

We restrict our attention to linear ranking functions, that is, a ranking function R such that

$R(k\tilde{a} + \tilde{b}) = kR(\tilde{a}) + R(\tilde{b})$ (3.4) for any $\tilde{a}$ and $\tilde{b}$ belonging to $F(\mathbb{R})$ and any $k \in \mathbb{R}$ (see in (Mahdavi-Amiri and Nasseri, 2007)).

Remark 3.1:

For any trapezoidal fuzzy number $\tilde{a}$, the relation $\tilde{a} \succeq \tilde{0}$ holds, if there exist $\varepsilon \geq 0$ and $\alpha \geq 0$ such that $\tilde{a} \geq (-\varepsilon, \varepsilon, \alpha, \alpha)$. We realize that $R(-\varepsilon, \varepsilon, \alpha, \alpha) = 0$ (we also consider $\tilde{a} \approx \tilde{0}$ if and only if $R(\tilde{a}) = 0$). Thus, without loss of generality, throughout the paper we let $\tilde{0} = (0,0,0,0)$ as the zero trapezoidal fuzzy number.

The following lemma is now immediately in hand.

Lemma 3.1:

Let R be any linear ranking function. Then,

(i) $\tilde{a} \succeq \tilde{b}$ If and only if $\tilde{a} - \tilde{b} \succeq \tilde{0}$ If and only if $-\tilde{b} \succeq -\tilde{a}$

(ii) If $\tilde{a} \succeq \tilde{b}$ and $\tilde{c} \succeq \tilde{d}$, then $\tilde{a} + \tilde{c} - \succeq \tilde{b} + \tilde{d}$.

We consider the linear ranking functions on $F(\mathbb{R})$ as:

$$R(\tilde{a}) = c_L a^L + c_U a^U + c_\alpha \alpha + c_\beta \beta, \tag{3.5}$$

where $\tilde{a} = (a^L, a^U, \alpha, \beta)$, and $c_L, c_U, c_\alpha, c_\beta$ are constants, at least one of which is nonzero. A special version of the above linear ranking function was first proposed by Yager (1994) (see also Fortemps and Roubens, 1996;Mahdavi-Amiri and Nasseri, 2007) as follows:

$$R(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf \tilde{a}_\lambda + \sup \tilde{a}_\lambda)\, d\lambda \tag{3.6}$$

which reduces to

$$R(\tilde{a}) = \frac{a^L + a^U}{2} + \frac{1}{4}(\beta - \alpha) \tag{3.7}$$

Then, for trapezoidal fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)$ ans $\tilde{b} = (b^L, b^U, \gamma, \theta)$ we have

$$\tilde{a} \succeq \tilde{b} \text{ if and only if } a^L + a^U + \frac{1}{2}(\beta - \alpha) \geq b^L + b^U + \frac{1}{2}(\theta - \gamma) \tag{3.8}$$

Quadratic Programming with Fuzzy Numbers:

A quadratic programming with fuzzy numbers (QPFN) problem is defined as:

$$\begin{aligned} &\textit{minimize } \frac{1}{2} x^T \tilde{Q} x + x^T \tilde{c} \\ &\textit{subject to } \tilde{A}x \approx \tilde{b} \end{aligned} \tag{4.1}$$

where $\tilde{b} \in (F(\mathbb{R}))^m, \tilde{Q} = [\tilde{q}_{ij}] \in (F(\mathbb{R}))^{n\times n}, \tilde{A} = [\tilde{a}_{ij}] \in (F(\mathbb{R}))^{m\times n}, \tilde{c} \in (F(\mathbb{R}))^n$ are given and $x \in R^n$ is to be determined.

The following theorem, which is an extension of the concepts concerning to the fuzzy arithmetic and linear ranking functions on quadratic programming with fuzzy numbers problems, will be useful for solving the QPFN problems.

Theorem 4.1:

The following quadratic programming problem and QPFN problem are equivalent.

$$\begin{aligned} &\textit{minimize } \frac{1}{2} x^T Q x + x^T c \\ &\textit{subject to } Ax = b \end{aligned} \tag{4.2}$$

where a_{ij}, q_{ij}, c_j and b_i, $i = 1,...,m, j = 1,...,n$ are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}, \tilde{q}_{ij}, \tilde{c}_{ij}$ and $\tilde{b}_i$, $i = 1,..., m, j = 1,..., n$ with respect to a given linear ranking function, respectively.

Proof. Let S_1 and S_2 be sets of all feasible solutions of (4.1) and (4.2), respectively. We first show that $S_1 = S_2$ and then prove that the optimal solution of the both problems is equal. We know that x $\in S_1$ if and only if $\tilde{A}x \approx \tilde{b}$ if and only if

$$R(\sum_{j=1}^{n} \tilde{a}_{ij} x_j) = R(\tilde{b}_i),\ i = 1,...,m,$$

if and only if

$$\sum_{j=1}^{n} R(\tilde{a}_{ij} x_j) = R(\tilde{b}_i),\ i = 1,...,m,$$

if and only if

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j = b_i,\ i = 1,...,m,$$

or

$$Ax = b,$$

where $a_{ij} = R(\tilde{a}_{ij}), b_i = R(\tilde{b}_i),$ for all $j = 1, ...,n$, and $i = 1, ...,m$, if and only if $x \in S_2$. Hence $S_1 = S_2$. So, the proof will be complete by this fact that for each linear ranking function R, we have

$$R(\frac{1}{2} x^T \tilde{Q} x + x^T \tilde{c}) = \frac{1}{2} x^T R(\tilde{Q} x) + R(x^T \tilde{c}) = \frac{1}{2} x^T R(\tilde{Q}) x + x^T R(\tilde{c}) = \frac{1}{2} x^T Q x + x^T c,$$

where $q_{ij} = R(\tilde{q}_{ij})$ and $c_j = R(\tilde{c}_j),$ for all $i,j=1,...,n$.

The mentioned theorem represent that for solving a QPFN problem it is enough to solve an equivalent quadratic programming in crisp environment. So here we deal with on these problems.

It is simple to show the QPFN problem has a unique solution if the matrix A is of full rank and the matrix Q is positive definite on the subspace $M = \{x \mid Ax = 0\}$. Now consider the Lagrange necessary conditions for this problem as follows:

$$\begin{aligned} Qx + A^T \lambda + c &= 0 \\ Ax \qquad - b &= 0 \end{aligned} \tag{4.3}$$

These correspond to the general condition Lagrange (see in (Bazaraa *et al.,* 2006)), and in this case they comprise an $(n + m)$ -dimensional linear system of equations. A natural question is whether the system is nonsingular. The following Theorem shows that the system is indeed nonsingular under the conditions stated above.

Theorem 4.2:

Let Q and A be $n \times n$ and $m \times n$ matrices, respectively. Suppose that A has rank m and that Q is positive definite on the subspace $M = \{x \mid Ax = 0\}$.
Then the matrix

$$\begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix}$$

is nonsingular

Proof. Suppose (x, y) $\in \mathbb{R}^{n+m}$ is such that

$$\begin{aligned} Qx + A^T y &= 0 \\ Ax \quad &= 0 \end{aligned} \tag{4.4}$$

Multiplication of the first equation by x^T yields

$$x^T Qx + x^T A^T y = 0$$

and substitution of $Ax = 0$ yields $x^T Qx = 0$. However, clearly $x \in M$, and thus the hypothesis on Q together with $x^T Qx = 0$ implies that $x = 0$. It then follows that the first equation that $A^T y = 0$. The full rank condition on A then implies that $y = 0$. Thus the only solution to (4.4) is $x = 0, y = 0$. □

Under the assumptions of Theorem 4.2, there are several methods for solving the system (4.3). As a general rules it is most efficient to use factorization methods (such as *LU* decomposition) that exploit the structure of the solving linear equations can be used.

Remark 4.1:

Note that if, as is often the case, the matrix Q is actually positive definite (over the whole space), then an explicit formula for the solution of the system can be easily derived as follows:
From the first equation in (4.3) we have

$$x = -Q^{-1} A^T \lambda - Q^{-1} c.$$

Substitution of this into the second equation then yields

$$-AQ^{-1} A^T \lambda - AQ^{-1} c - b = 0.$$

from which we immediately obtain

$$\lambda = -(AQ^{-1} A^T)^{-1} [AQ^{-1} c + b]$$

and

$$\begin{aligned} x &= -Q^{-1} A^T (AQ^{-1} A^T)^{-1} [AQ^{-1} c + b] - Q^{-1} c \\ &= -Q^{-1} [I - A^T (AQ^{-1} A^T)^{-1} AQ^{-1}] c + Q^{-1} A^T (AQ^{-1} A^T)^{-1} b. \end{aligned} \tag{4.5}$$

This representation is useful in theoretical developments, although in practice the solution may be calculated by some other method as discussed above.

Conclusions:

Using the linear ranking functions is a convenient approach for ordering fuzzy numbers. We defined the fuzzy version of the quadratic programming problems and discussed how can solve them, by using linear ranking functions and also Iranian National Elite Foundation.

ACKNOWLEDGMENTS

This work was supported in part by the Research Center of Algebraic Hyper-structures and Fuzzy Mathematics, and alao Iranian National Elite Foundation.

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