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Numerical Solutions for the Complex Nonlinear Wave Equation by the Homotopy Perturbation Method

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Abstract: In this paper, the homotopy perturbation method is extended to investigate the numerical solutions of the complex nonlinear wave equation. To examine the accuracy of the method, the available analytical solutions of the coupled nonlinear differential equations are compared with the homotopy perturbation method. The numerical results validate the convergence and accuracy of the homotopy perturbation method. Finally, the accuracy properties are demonstrated by some examples.

Key words: Homotopy perturbation method, Nonlinear wave equation

INTRODUCTION

To find the explicit solutions of nonlinear differential equations, many powerful methods have been used (Abbasbandy, 2006)-(Liao, 1995). The homotopy perturbation method (HPM) (He, 1998), proposed first by He in 1998. The applications of the HPM (Abbasbandy, 2006)-(Ghasemi, *et.al*., 2007) have appeared in a lot of research, especially during recent years, showing that the method is a powerful technique for studying the numerical solutions. The HPM always continuously deforms a simple problem which is easy to solve into the difficult problem under study. With this method, the rapidly convergent series solutions can be obtained, along with easily computable components.

One of the most important problems in mathematical physics is the vibration of a stretched string. Simplicity and frequent occurrence in many branches of mathematical physics make it a classic example in the theory of partial differential equations.

We would like to extend the applications of the HPM from the single equation to the coupled systems to construct the numerical solutions for the complex nonlinear wave equation (NWE).

The complex NWE is written as:

$$\begin{cases} \frac{\partial^2 U}{\partial x^2} - U\frac{\partial^2 U}{\partial t^2} = \phi(x,t), \quad 0 \le x,t \le 1/2 \\ U(0,t) = f(t), \quad \frac{\partial}{\partial x}U(0,t) = g(t), \end{cases} \tag{1}$$

where $U = u + Iv,\ \phi = \phi_1 + I\phi_2,\ f = f_1 + If_2,\ g = g_1 + Ig_2,\ I = \sqrt{-1}.$

In fact, the study of the complex NWE is very interesting and important because when substituting $U=u+iv$ into Eq. (1), collecting the real and imaginary parts, a coupled system

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \left(u\frac{\partial^2 u}{\partial t^2} - v\frac{\partial^2 v}{\partial t^2}\right) = \phi_1(x,t), \quad 0 \le x,t \le 1/2 \\ \frac{\partial^2 v}{\partial x^2} - \left(u\frac{\partial^2 v}{\partial t^2} + v\frac{\partial^2 u}{\partial t^2}\right) = \phi_2(x,t), \end{cases} \tag{2}$$

is derived. In other words, the two Eqs. (1) and (2) are equal.

The paper is organized as follows: In Section 2, some descriptions are given on the HPM to the coupled systems. In Section 3, we present the numerical solutions by some examples and show the efficiency and simplicity of the proposed method. Finally, the sections are followed by the conclusion.

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2 The Homotopy Perturbation Method to the Coupled Systems:

The HPM to the coupled systems is described as follows. Consider the nonlinear coupled differential equations in this form:

$$\begin{cases} L_1(u) + R_1(u,v) - f_1(r) = 0, \\ L_2(v) + R_2(u,v) - f_2(r) = 0, \quad r \in \Omega \end{cases} \tag{3}$$

with the boundary conditions of

$$B_1\left(u, \frac{\partial u}{\partial \mathbf{n}}\right) = 0, \qquad B_2\left(v, \frac{\partial v}{\partial \mathbf{n}}\right) = 0, \qquad r \in \Gamma$$

where Li (i=1,2) are the linear operators, and their inverse operator can be easily solved; R_i are the operators for the remaining parts of the functions u, v; $f_i(r)$ are the known analytical functions, B_i are the boundary operators, Γ is the boundary of the domain Ω, and $\mathbf{n}$ is the unit outward normal of Ω. We construct the following homotopy $\Omega \times [0, 1] \to R$ for the extended HPM, which satisfies

$$\begin{cases} H_1(u,v,p) = (1-p)[L_1(u) - L_1(u_0)] + p[L_1(u) + R_1(u,v) - f_1(r)] = 0, \\ H_2(u,v,p) = (1-p)[L_2(v) - L_2(v_0)] + p[L_2(v) + R_2(u,v) - f_2(r)] = 0. \end{cases} \tag{4}$$

Or

$$\begin{cases} H_1(u,v,p) = L_1(u) - L_1(u_0) + p[L_1(u_0) + R_1(u,v) - f_1(r)] = 0, \\ H_2(u,v,p) = L_2(v) - L_2(v_0) + p[L_2(v_0) + R_2(u,v) - f_2(r)] = 0, \end{cases} \tag{5}$$

where $p \in [0,1]$ is an embedding parameter; u_0 and v_0 are the known initial approximations of Eq. (3) that satisfy the boundary conditions mentioned above. It is clear that

$$\begin{cases} H_1(u,v,0) = L_1(u) - L_1(u_0) = 0, \\ H_2(u,v,0) = L_2(v) - L_2(v_0) = 0, \\ H_1(u,v,1) = L_1(u) + R_1(u,v) - f_1(r) = 0, \\ H_2(u,v,1) = L_2(v) + R_2(u,v) - f_2(r) = 0. \end{cases} \tag{6}$$

This shows that $H_1(u,v,p)$ and $H_2(u,v,p)$ continuously deform from the trivial problems $L_1(u)$-$L_1(u_0)$=0 and $L_2(v)$-$L_2(v_0)$=0 respectively, to the original problem investigated (3). In topology, the deformations from $L_1(u)$-$L_1(u_0)$=0 to $L_1(u)+R_1(u,v)-f_1(r)$ and from $L_2(u)$-$L_2(u_0)$=0 to $L_2(u)+R_2(u,v)-f_2(r)$, are known to be homotopic.

For the coupled system (3), we assume that the solutions $u(x,t)$ and $v(x,t)$ are given by the infinity series of the homotopy parameter p in the form

$$\begin{cases} u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots \\ v = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + \cdots \end{cases} \tag{7}$$

The approximate solutions of (3) can be obtained when $p \to 1$:

$$\begin{cases} u = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i u_i = u_0 + u_1 + u_2 + \cdots \\ v = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i v_i = v_0 + v_1 + v_2 + \cdots \end{cases} \tag{8}$$

The convergence must depend on the choices for L_i R_i and f_i (i=1,2) and the initial conditions. The series (7), is convergent for most cases (He, 1998). In the following section, we will apply the HPM to the complex NWE equation to derive the numerical solutions.

3 Numerical Examples and Comparisons:

Consider the operator form of the real and imaginary parts of the complex NWE (1)

$$\begin{cases} L_1(u)-(uv_{tt}-vv_{tt})-\phi_1=0 \\ L_2(v)-(uv_{tt}-vu_{tt})-\phi_2=0 \end{cases} \tag{9}$$

where the linear operator $L_1 = L_2 = L = \frac{\partial^2}{\partial x^2}$ with the inverse operator. $L^{-1} = \int_0^x\int_0^x \mathrm{d}x\mathrm{d}x$ Applying the inverse operator, and according to the extended HPM, we construct the following homotopy:

$$\begin{cases} H_1(u,v,p) = u(x,t) - h_1(x,t) - p\int_0^x\int_0^x \left(u\frac{\partial^2 u}{\partial t^2} - v\frac{\partial^2 v}{\partial t^2}\right)\mathrm{d}x\mathrm{d}x = 0, \\ H_2(u,v,p) = v(x,t) - h_2(x,t) - p\int_0^x\int_0^x \left(u\frac{\partial^2 v}{\partial t^2} + v\frac{\partial^2 u}{\partial t^2}\right)\mathrm{d}x\mathrm{d}x = 0, \end{cases} \tag{10}$$

where

$$h_i(x,t) = f_i(t) + xg_i(t) + \int_0^x\int_0^x \phi_i(x,t)\mathrm{d}x\mathrm{d}x, \quad i = 1,2.$$

Substituting (7) into (10), and equating the terms with identical powers of p, we find

$$p^0: u_0(x,t) = h_1(x,t),$$

$$v_0(x,t) = h_2(x,t),$$

$$p^1: u_1 = \int_0^x\int_0^x \left(u_0\frac{\partial^2 u_0}{\partial t^2} - v_0\frac{\partial^2 v_0}{\partial t^2}\right)\mathrm{d}x\mathrm{d}x,$$

$$v_1 = \int_0^x\int_0^x \left(u_0\frac{\partial^2 v_0}{\partial t^2} + v_0\frac{\partial^2 u_0}{\partial t^2}\right)\mathrm{d}x\mathrm{d}x,$$

$$p^2: u_2 = \int_0^x\int_0^x \left(u_0\frac{\partial^2 u_1}{\partial t^2} + u_1\frac{\partial^2 u_0}{\partial t^2} - v_0\frac{\partial^2 v_1}{\partial t^2} - v_1\frac{\partial^2 v_0}{\partial t^2}\right)\mathrm{d}x\mathrm{d}x,$$

$$v_2 = \int_0^x\int_0^x \left(u_0\frac{\partial^2 v_1}{\partial t^2} + u_1\frac{\partial^2 v_0}{\partial t^2} + v_0\frac{\partial^2 u_1}{\partial t^2} + v_1\frac{\partial^2 u_0}{\partial t^2}\right)\mathrm{d}x\mathrm{d}x,$$

$\vdots$

We can obtain the approximation solution to the real and imaginary parts of the complex NWE (1) in finite series as follows :

$$u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \cdots, \qquad v = \sum_{i=0}^{\infty} v_i = v_0 + v_1 + v_2 + \cdots$$

By computing some terms, say $k, u \simeq u_a = \sum_{i=0}^{\infty} u_i$, and $v \simeq v_a = \sum_{i=0}^{\infty} v_i$, an approximate solution would be achieved. To give a clear overview of the methodology, as an illustrative example, we take the complex nonlinear wave equation described by

$$\begin{cases} \frac{\partial^2}{\partial x^2}U - U\frac{\partial^2}{\partial t^2}U = 1 + 3/2(x^2 + t^2) + 2I(1 - x^2 - t^2), \ 0 \le x,t \le 1/2 \\ U(0,t) = \frac{t^2}{2} + It^2, \ \frac{\partial}{\partial x}U(0,t) = 0, \end{cases} \tag{11}$$

where $U = u + Iv, I = \sqrt{-1}$.

Substituting (7) into (10) yields a system of algebraic equations of p. Through collecting all the same power of p, and setting all the coefficients of the terms p^i to zero, we can get an over-determined system of differential equations with unknown variables u_i and v_i (i=0,1,...). For the convenience of the readers, we only write the first few terms of equations. Also, with the aid of the symbolic computation system Maple, we can easily obtain the solution:

$$p^0:\ u_0 = 1/2\,t^2 + 1/8\,x^4 + 1/2\,x^2 + 3/4x^2t^2,$$

$$v_0 = t^2 - 1/6\,x^4 + x^2 - x^2t^2,$$

$$p^1:\ u_1 = -\frac{1}{384}x^8 + \frac{77}{720}x^6 - \frac{7}{240}x^6t^2 + \frac{11}{24}x^4t^2 - 1/8\,x^4 - 3/4\,x^2t^2,$$

$$v_1 = -\frac{1}{112}x^8 + \frac{77}{360}x^6 - \frac{1}{10}x^6t^2 + \frac{1}{12}x^4t^2 + \frac{1}{6}x^4 + x^2t^2$$

$$p^2:\ u_2 = \frac{13494}{53222400}x^{12} - \frac{527}{15724800}x^{14}t^2 - \frac{4321}{79833600}x^{12}t^2 + \frac{299}{201600}x^{10}t^2$$

$$+\frac{779}{725760}x^{10} - \frac{527}{182476800}x^{16} - \frac{7453}{1037836800}x^{14} + \frac{49}{1440}x^6t^2 - \frac{11}{48}x^4t^2 -$$

$$\frac{11}{120}x^6 + \frac{13}{1920}x^8 + \frac{533}{80640}x^8t^2,$$

$$v_2 = \frac{857}{14968800}x^{12} - \frac{223291}{7264857600}x^{14}t^2 - \frac{439}{2494800}x^{12}t^2 + \frac{527}{12758777856000}x^{24} +$$

$$\frac{2304613}{18584668118016000}x^{22} + \frac{29}{113400}x^{10}t^2 - \frac{1913}{1814400}x^{10} + \frac{31}{43589145600}x^{22}t^2 -$$

$$\frac{3089579}{871782912000}x^{16} - \frac{202457}{10897286400}x^{14} + \frac{36492997}{266765571072000}x^{18}t^2 - \frac{10715}{27897053184}x^{16}t^2 +$$

$$\frac{61947029}{9654373048320000}x^{20} - \frac{12257249}{533531142144000}x^{18} + \frac{7}{60}x^6t^2 - \frac{1}{24}x^4t^2 - \frac{1}{60}x^6 +$$

$$\frac{13}{560}x^8 + \frac{118733}{94650716160000}x^{20}t^2 - \frac{53}{8064}x^8t^2,$$

$$\vdots$$

Hence, we obtain an approximate solution of the complex NWE as follows:

$$u(x,t) \simeq -\frac{527}{91238400}x^{16} - \frac{7453}{518918400}x^{14} + \frac{1}{8}x^4 + \frac{1}{2}t^2 + \frac{49}{720}x^6t^2 - \frac{11}{24}x^4t^2 +$$

$$\frac{1}{2}x^2 + 3/4\ x^2t^2 + \frac{13}{960}x^8 - \frac{11}{60}x^6 + \frac{13949}{26611200}x^{12} + \frac{799}{362880}x^{10} + \frac{299}{100800}x^{10}t^2 +$$

$$\frac{533}{40320}x^8t^2 - \frac{527}{7862400}x^{14}t^2 - \frac{4321}{39916800}x^{12}t^2,$$

$$v(x,t) \simeq -\frac{3089579}{435891456000}x^{16} - \frac{202457}{5448643200}x^{14} + \frac{36492997}{133382785536000}x^{18}t^2$$

$$-\frac{10715}{13948526592}x^{16}t^2-\frac{1}{6}x^4+t^2+\frac{527}{6379388928000}x^{24}+\frac{7}{30}x^6t^2-\frac{1}{12}x^4t^2$$

$$+\frac{61947029}{4827186524160000}x^{20}-\frac{12257249}{266765571072000}x^{18}+\frac{2304613}{929233405900800 0}x^{22}$$

$$+\frac{13}{280}x^8-\frac{1}{30}x^6+\frac{31}{21794572800}x^{22}t^2+\frac{118733}{47325358080000}x^{20}t^2+\frac{857}{7484400}x^{12}$$

$$-\frac{1913}{907200}x^{10}+\frac{29}{56700}x^{10}t^2-\frac{53}{4032}x^8t^2+x^2-x^2t^2-\frac{223291}{3632428800}x^{14}t^2$$

$$-\frac{439}{1247400}x^{12}t^2.$$

Continuing this process, the complex solution

$$u=\lim_{n\to\infty}\sum_{i=0}^{n}u_i\,,\quad v=\lim_{n\to\infty}\sum_{i=0}^{n}v_i$$

is found by means of n-term approximation. For this example, we consider the three-terms approximation $u\simeq u_a=u_0+u_1+u_2,\ v\simeq v_a=v_0+v_1+v_2$. Continuing this technique, the analytical solution is obtained $u(x,t)=(x^2+t^2)/2,\ v(x,t)=x^2+t^2$

The error, the exact solution, and the numerical results are given and compared to each other in Table 1 for some values of x and t, where

$$Error=|(u+Iv)-(u_a+Iv_a)|=\sqrt{(u-u_a)^2+(v-v_a)^2}.$$

As expected, the numerical solutions in Table 1 clearly indicate how the decomposition scheme obtains efficient results much closer to the actual solutions. At this point, we consider the following nonlinear wave equation (Ghasemi *et al.,* 2007)

$$\frac{\partial^2 u}{\partial x^2}-u\frac{\partial^2 u}{\partial t^2}=1-(x^2+t^2)/2,\qquad 0\le x,t\le 1, \tag{12}$$

with initial conditions

$u(0,t)=\frac{t^2}{2},\ \frac{\partial}{\partial x}u(0,t)=0$

The exact solution of (12), for the special case of $v=f_2=g_2=\phi_2=0$, is given by $u(x,t)=\frac{x^2+t^2}{2}$.

In this case, we have the following single homotopy form from (10).

$$H_1(u,v,p)=u(x,t)-h_1(x,t)-p\int_0^x\int_0^x u\frac{\partial^2 u}{\partial t^2}\,\mathrm{d}x\mathrm{d}x=0 \tag{13}$$

where $h_1(x,t)=f_1(t)+\int_0^x\int_0^x\phi_1(x,t)\,\mathrm{d}x\mathrm{d}x$.

Substituting (7) into (13), and equating the terms with identical powers of p, we will find the approximation solution which is denoted in (Ghasemi *et al.,* 2007).

Table 1: Comparison of the exact solution with the numerical solution

x	t	u	v	u_a	v_a	Error
0.00	0.00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
0.02	0.02	6.250E-04	1.250E-03	6.253E-04	1.250E-03	5.696E-07
0.05	0.05	2.500E-03	5.000E-03	2.505E-03	4.993E-03	9.110E-06
0.08	0.08	5.625E-03	1.125E-02	5.653E-03	1.121E-02	4.609E-05
0.10	0.10	1.000E-02	2.000E-02	1.009E-02	1.988E-02	1.455E-04
0.12	0.12	1.562E-02	3.125E-02	1.584E-02	3.096E-02	3.549E-04
0.15	0.15	2.250E-02	4.500E-02	2.294E-02	4.441E-02	7.349E-04
0.18	0.18	3.062E-02	6.125E-02	3.143E-02	6.015E-02	1.359E-03
0.20	0.20	4.000E-02	8.000E-02	4.136E-02	7.813E-02	2.315E-03
0.22	0.22	5.062E-02	1.012E-01	5.278E-02	9.825E-02	3.699E-03
0.25	0.25	6.250E-02	1.250E-01	6.576E-02	1.204E-01	5.624E-03
0.28	0.28	7.562E-02	1.512E-01	8.035E-02	1.445E-01	8.212E-03
0.30	0.30	9.000E-02	1.800E-01	9.663E-02	1.705E-01	1.160E-02
0.32	0.32	1.056E-01	2.112E-01	1.146E-01	1.981E-01	1.592E-02
0.35	0.35	1.225E-01	2.450E-01	1.345E-01	2.273E-01	2.133E-02
0.38	0.38	1.406E-01	2.812E-01	1.562E-01	2.580E-01	2.800E-02
0.40	0.40	1.600E-01	3.200E-01	1.798E-01	2.898E-01	3.610E-02
0.42	0.42	1.806E-01	3.612E-01	2.055E-01	3.228E-01	4.579E-02
0.45	0.45	2.025E-01	4.050E-01	2.332E-01	3.567E-01	5.727E-02
0.48	0.48	2.256E-01	4.512E-01	2.630E-01	3.912E-01	7.071E-02
0.50	0.50	2.500E-01	5.000E-01	2.950E-01	4.263E-01	8.632E-02

Conclusion:

In this study, we successfully apply the homotopy perturbation method to approximate the complex solution of a nonlinear wave equation. It gives a simple and a powerful mathematical tool for nonlinear problems. In our study, we use the *Maple* Package to calculate the series obtained from the iteration method.

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