Exact Soliton Solutions for Equal Width Wave Equation

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Abstract: The homogeneous balance method is used to construct exact traveling wave solutions of equal width wave equation, in which the homogeneous balance method is applied to solve the Riccati equation and the reduced nonlinear ordinary differential equation, respectively. Many exact traveling wave solutions of equal width wave equation are successfully obtained, which contain soliton-like and periodic-like solutions. This method is straightforward and concise, and it can also be applied to other nonlinear evolution equations.

Key words: Homogeneous balance method; Equal width wave equation; Riccati equation; Soliton-like solution; Periodic-like solution

INTRODUCTION

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years Wang (1995; 1996), Khalafallah (2009) used a useful homogeneous balance (HB) method for finding exact solutions of a given nonlinear partial differential equations. Fan (2000) used HB method to search for Backlund transformation and similarity reduction of nonlinear partial differential equations. In this paper, we use the homogeneous balance method (HB) to solve the Riccati equation \[ \phi' = a\phi^2 + b\phi + c \] and the reduced nonlinear ordinary differential equation for equal width wave equation, respectively. It makes the HB method use more extensively.

Equal Width Wave Equation:

For equal width wave equation (Evans and Raslan, 2005)

\[ u_t + \varepsilon uu_x - vu_{xxt} = 0, \]  \hspace{1cm} (1)

where $\varepsilon$ and $v$ are nonnegative constants.

Let us consider the traveling wave solutions

\[ u(x,t) = u(\xi), \xi = kx + lt + \xi_0 \] \hspace{1cm} (2)

Where $k,l$ and $\xi_0$ are constants.

Substituting (2) into (1), then (1) is reduced to the following nonlinear ordinary differential equation

\[ lu' + \varepsilon ku u' - v k^2 lu''' = 0, \] \hspace{1cm} (3)

We now look for the solutions of (3) in the form
\[ u = \sum_{i=0}^{m} a_i \phi^i, \]  
(4)

where \( a_i \) are constants to be determined later and \( \phi \) satisfy the Riccati equation

\[ \phi' = a\phi^2 + b\phi + c \]  
(5)

where \( a, b \) and \( c \) are constants.

Balancing \( u^{'''} \) with \( uu' \) in (3) gives

\[ m + 3 = m + m + 1, \]

does not hold.

so that

\[ m = 2. \]

The homogeneous balance method admits the use of the finite expansion in the form

\[ u = a_0 + a_1 \phi + a_2 \phi^2, \]  
(6)

where \( a_0, a_1 \) and \( a_2 \) are constants to be determined and \( \phi \) satisfy Eq. (5).

Substituting Eqs. (6) and (5) into Eq. (3), and equating the coefficients of like powers of \( \phi(i = 0, 1, 2, 3, 4, 5) \) to zero yields a set of algebraic system for \( a_0, a_1, a_2, k \) and \( l \)

\[
\begin{align*}
la_1c - vk^2l(a_1(2ac^2 + b^2c) + 6a_2c^2b) + \varepsilon ka_0a_1c &= 0, \\
l(a_1 + 2a_2c) - vk^2l(6a_2c(2ac + b^2) + 6a_2b^2c) + 2a_2(2ac^2 + b^2c) + a_1(b(2ac + b^2) + 6abc)) + \varepsilon ka_0(a_1 + 2a_2c) + \varepsilon ka_1^2c &= 0, \\
l(a_1 + 2a_2b) - vk^2l(2a_2(b(2ac + b^2) + 6abc) + a_1(3b^2a + 4a(2ac + b^2)) + 2\varepsilon a_2(ab + 6a_2b(2ac + b^2))) + \varepsilon ka_0(aa_1 + 2a_2b) + \varepsilon ka_1(a_1b + 2a_2c) + \varepsilon ka_2a_2c &= 0, \\
2laa_2 - vk^2l(12a_2ca^2 + 18a_2b^2a + 6aa_2(2ac + b^2)) + 2a_2(3b^2a + 4a(2ac + b^2)) + 12a_2a^2b + 2\varepsilon a_0a_2a + \varepsilon ka_1(a_1b + 2a_2c) + \varepsilon ka_2(2a_2b) - vk^2l(54a_2ba^2 + 6a_3a_1) &= 0, \\
2\varepsilon ka_2a_2 - 24vk^2a_1a_2 &= 0.
\end{align*}
\]  
(7)

For which, with the aid of Maple, we find

\begin{align*}
a_0 &= \frac{l(8vk^2ac + b^2v^2 - 1)}{\varepsilon k}, \\
a_1 &= \frac{12vkab}{\varepsilon}, \\
a_2 &= \frac{12vkla^2}{\varepsilon}.
\end{align*}

(8)
It is to be noted that the Riccati equation (5) can be solved using the homogeneous balance method as follows

**Case I.** Let \( \phi = \sum_{i=0}^{m} b_i \tanh^i \xi \). Balancing \( \phi' \) with \( \phi^2 \) leads to

\[
\phi = b_0 + b_1 \tanh \xi. 
\]  
(9)

Substituting Eq. (9) into Eq. (5) we obtain the following solution of Eq. (5)

\[
\phi = \frac{-1}{2a} (b + 2 \tanh \xi), \quad ac = \frac{b^2}{4} - 1. 
\]  
(10)

Substituting Eqs. (8) and (10) into (2) and (6), we get the following traveling wave solutions of equal width wave equation (1):

\[
u(x,t) = -l + 8 v k^2 a c l + b^2 v k^2 - 3 v k^2 b^2 \frac{2}{\varepsilon} \tanh^2 (kx + lt + \xi_0),
\]  
(11)

where \( ac = \frac{b^2}{4} - 1. \)

Similarly, let \( \phi = \sum_{i=0}^{m} b_i \coth^i \xi \), then we obtain the following new traveling wave soliton solutions of equal width wave equation (1):

\[
u(x,t) = -l + 8 v k^2 a c l + b^2 v k^2 - 3 v k^2 b^2 \frac{2}{\varepsilon} \coth^2 (kx + lt + \xi_0),
\]  
(12)

where \( ac = \frac{b^2}{4} - 1. \)

**Case II.** From (Zhao and Tang, 2002), when \( a=1, b=0 \) the Riccati Eq. (5) has the following solutions

\[
\phi = -\sqrt{-c} \tanh (\sqrt{-c} \xi), \quad c < 0,
\]

\[
\phi = -\frac{1}{\varepsilon}, \quad c = 0,
\]

\[
\phi = \sqrt{c} \tanh (\sqrt{c} \xi), \quad c > 0.
\]

From (6), (8) and (13), we have the following new traveling wave solutions of equal width wave equation (1) which contain traveling wave solutions, periodic wave solutions and rational solutions as follows

When \( c < 0 \), we have the following traveling wave (soliton-like) solutions of equal width wave equation (1):

\[
u(x,t) = \frac{l(8 v k^2 a c + b^2 v k^2 - 1)}{\varepsilon k} - \frac{12 v k l a b}{\varepsilon} \tanh (\sqrt{-c} (kx + lt + \xi_0)) - \frac{12 v k l a ^2 c}{\varepsilon} \tanh^2 (\sqrt{-c} (kx + lt + \xi_0)).
\]  
(14)

When \( c = 0 \) we have

\[
u(x,t) = \frac{l(8 v k^2 a c + b^2 v k^2 - 1)}{\varepsilon k} - \frac{12 v k l a b}{\varepsilon (kx + lt + \xi_0)} + \frac{12 v k l a ^2}{\varepsilon (kx + lt + \xi_0)^2}.
\]  
(15)
When \( c > 0 \) we have the following traveling wave (periodic-like) solutions of equal width wave equation (1):

\[
\begin{align*}
\phi & = A_0 + \sum_{i=1}^{\infty} (A_i f^i + B_i f^{i-1} g), \\
& \text{with} \\
f & = \frac{1}{\cosh \xi + r}, \\
g & = \frac{\sinh \xi}{\cosh \xi + r}.
\end{align*}
\]

Balancing \( \phi' \) with \( \phi^2 \) leads to

\[
\phi = A_0 + A_1 f + B_1 g.
\]

Substituting Eq. (18) into (5), collecting the coefficient of the same power \( f^i(\xi)g^j(\xi)(i = 0, 1, 2; j = 0, 1) \) and setting each of the obtained coefficients to zero yield the following set of algebra equations

\[
\begin{align*}
2A_0 & = 0, \\
2A_0 A_i & - 2aB_i - rB_i + bA_i = 0, \\
aA_i^2 & + a(r^2 - 1)B_i^2 + (r^2 - 1)B_i = 0, \\
2aA_i B_i & + bB_i = 0, \\
2aA_i B_i & + A_i = 0,
\end{align*}
\]

which have solutions

\[
A_0 = -\frac{b}{2a}, \quad A_i = \pm \sqrt{\frac{(r^2 - 1)}{4a^2}}, \quad B_i = -\frac{1}{2a}c = \frac{b^2 - 1}{4a}.
\]

From Eqs. (18), (19), we obtain

\[
\phi = -\frac{1}{2a} \left( b + \frac{\sinh \xi \pm \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right).
\]

From Eqs. (6), (8), and (20), we obtain the new solutions of equal width wave equation

\[
\begin{align*}
\phi & = e^{\xi} \phi(z) + p_\alpha(\xi),
\end{align*}
\]
where
\[ z = e^{p_3} \rho(z) + p_3, \]
where \( p_1, p_2 \) and \( p_3 \) are constants to be determined.

Substituting (22) into (5) we find that when \( c = \frac{-p_1^2 + b^2}{4a} \), we have
\[
\phi = -\frac{p_1 e^{p_3}}{a(e^{p_3} + p_3)} + \frac{p_1 - b}{2a}. \tag{23}
\]

If \( p_3 = 1 \) in (23), we have
\[
\phi = -\frac{p_1}{2a} \tanh\left(\frac{1}{2} p_3 \xi\right) - \frac{b}{2a}. \tag{24}
\]

If \( p_3 \) in (23), we have
\[
\phi = -\frac{p_1}{2a} \coth\left(\frac{1}{2} p_3 \xi\right) - \frac{b}{2a}. \tag{25}
\]

From (6), (8) and (23), we obtain the following new traveling wave solutions of equal width wave equation (1):
\[
u(x,t) = \frac{l(8v^2 v_c + b^2 v_c^2 - 1) + 12v k l}{e k} \frac{\phi e^{p_3(kx + lt + \xi_0)} - p_3 - b}{2}
\times \left(\frac{p_1 e^{p_3(kx + lt + \xi_0)}}{e^{p_3(kx + lt + \xi_0)} + p_3}\right). \tag{26}
\]

When \( p_3 = 1 \) we have the following traveling wave (soliton-like) solutions of equal width wave equation (1):
\[
u(x,t) = \frac{-l + 8v^2 v_c + b^2 v_c^2 - 3v k l b^2 + \frac{3v k l p_1^2}{e} \tanh\left(\frac{1}{2} p_1(kx + lt + \xi_0)\right)}. \tag{27}
\]

When \( p_3 = -1 \) we have the following traveling wave (periodic-like) solutions of equal width wave equation (1):
\[
u(x,t) = \frac{-l + 8v^2 v_c + b^2 v_c^2 - 3v k l b^2 + \frac{3v k l p_1^2}{e} \coth\left(\frac{1}{2} p_1(kx + lt + \xi_0)\right)}. \tag{28}
\]

Case: V. We suppose that the Riccati equation (5) have the following solutions of the form:
\[
\phi = A_0 + \sum_{i=1}^{m} \sinh^{i-1}(A_i \sinh w + B_i \cosh w), \tag{29}
\]
where \( \frac{d\phi}{d\xi} = \sinh w \) or \( \frac{d\phi}{d\xi} = \cosh w \). It is easy to find that \( m = 1 \) by balancing \( \phi' \) and \( \phi^2 \).

So we choose
\[
\phi = A_0 + A_1 \sinh w + B_1 \cosh w, \tag{30}
\]
when \( \frac{d\phi}{d\xi} = \sinh w \), we substitute (30) and \( \frac{d\phi}{d\xi} = \sinh w \), into (5) and set the coefficients of
\[
sinh' w \cosh' w(i = 0, 1, 2; j = 0, 1) \text{ to zero. A set of algebraic equations is obtained as follows:}
\]
\[
aA_0^2 + aB_1^2 + bA_i + c = 0,
\]
\[
2aA_0A_i + bA_i = 0,
\]
\[
aA_0^2 + aB_i^2 - B_i = 0,
\]
\[
2aA_0B_i + bB_i = 0,
\]
\[
2aA_iB_i + A_i = 0,
\]
for which, we have the following solutions:
\[
A_0 = -\frac{b}{2a}, \quad A_i = 0, \quad B_i = \frac{1}{a}, \quad (31)
\]
where \( c = \frac{b^2 - 4}{4a} \), and
\[
A_0 = -\frac{b}{2a}, \quad A_i = \pm \frac{1}{2a} \quad B_i = \frac{1}{2a}, \quad (32)
\]
where \( c = \frac{b^2 - 1}{4a} \).

To \( \frac{dw}{d\xi} = \sinh w \), we have
\[
\sinh w = -\csc h\xi, \quad \cosh w = -\coth \xi. \quad (33)
\]

From (31) – (33), we obtain
\[
\phi = -\frac{b + 2\coth \xi}{2a}, \quad (34)
\]
where \( c = \frac{b^2 - 4}{4a} \), and
\[
\phi = -\frac{b \pm \csc h\xi + \coth \xi}{2a}, \quad (35)
\]
where \( c = \frac{b^2 - 1}{4a} \).

From Eqs. (6), (8), (34) and (35) we get the following traveling wave solutions of equal width wave equation (1):
\[
u(x,t) = -\frac{l + 8vk^2 acl + lb^2vk^2 - 3vk^2lb^2}{ek} + \frac{12vlk}{e} - \coth^2(kx + lt + \xi_0), \quad (36)
\]
where \( c = \frac{b^2 - 4}{4a} \), and
where \( c = \frac{b^2 - 1}{4a} \).

Similarly, when \( \frac{dw}{d\xi} = \cosh w \), we obtain the following traveling wave (periodic-like) solutions of equal width wave equation (1):

\[
\begin{align*}
\frac{3ykl}{\epsilon} (\coth(kx + lt + \xi_0) \pm \csc(h(kx + lt + \xi_0)))^2,
\end{align*}
\]

\[
\begin{align*}
\frac{3ykl}{\epsilon} (\coth(kx + lt + \xi_0) \pm \csc(h(kx + lt + \xi_0)))^2,
\end{align*}
\]

where \( c = \frac{b^2 - 4}{4a} \), and

\[
\begin{align*}
u(x,t) = \frac{-l + 8vk^2 acl + lb^2vk^2 - 3vk^2lb^2}{\epsilon k} + \frac{12ykl}{\epsilon} \cot^2(kx + lt + \xi_0),
\end{align*}
\]

\[
\begin{align*}
u(x,t) = \frac{-l + 8vk^2 acl + lb^2vk^2 - 3vk^2lb^2}{\epsilon k} + \frac{12ykl}{\epsilon} \cot^2(kx + lt + \xi_0),
\end{align*}
\]

where \( c = \frac{b^2 - 1}{4a} \).

In summary we have used the extended homogeneous balance method to obtain many traveling wave solutions of equal width wave equation. We now summarize the key steps as follows:

**Step 1:** For a given nonlinear evolution equation

\[
\begin{align*}
F(u, u_t, u_x, u_{xt}, u_{tt}, \ldots) = 0,
\end{align*}
\]

we consider its traveling wave solutions \( u(x,t) = u(\xi), \xi = kx + lt + \xi_0 \) then Eq. (40) is reduced to an nonlinear ordinary differential equation

\[
\begin{align*}
Q(u, u', u'', u''', \ldots) = 0,
\end{align*}
\]

where a prime denotes \( \frac{d}{d\xi} \).

**Step 2:** For a given ansatz equation (for example, the ansatz equation is \( \phi' = a\phi^2 + b\phi + c \) in this paper), the form of \( u \) is decided and the homogeneous balance method is used on Eq. (41) to find the coefficients of \( u \).

**REFERENCES**
