

Generalization of first order differential operators associated to the Cauchy-Riemann operator in the R^n

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Abstract: In this paper we will discuss on the initial value problems of type

$$\frac{\partial w}{\partial t} = Fw, w(0, \cdot) = \phi$$

where t is the time and F is a linear first order operator acting in the

R^n . The article in hand constructs, conversely, all linear operators F for which the initial value problem with an arbitrary holomorphic initial function is always solvable.

Key words: Initial value problems of Cauchy Kovalevskaya type; associated differential operators

INTRODUCTION

In this paper we consider the linear first order systems

$$\begin{aligned} \frac{\partial u}{\partial t} &= a_{11} \frac{\partial u}{\partial x_1} + a_{12} \frac{\partial u}{\partial x_2} + \dots + a_{1n} \frac{\partial u}{\partial x_n} + a_{21} \frac{\partial u}{\partial y_1} + a_{22} \frac{\partial u}{\partial y_2} + \dots + a_{2n} \frac{\partial u}{\partial y_n} + a_{31} \frac{\partial v}{\partial x_1} + a_{32} \frac{\partial v}{\partial x_2} + \dots + a_{3n} \frac{\partial v}{\partial x_n} + \\ &a_{41} \frac{\partial v}{\partial y_1} + a_{42} \frac{\partial v}{\partial y_2} + \dots + a_{4n} \frac{\partial v}{\partial y_n} + c_1 u + c_2 v + c_3 \\ \frac{\partial v}{\partial t} &= b_{11} \frac{\partial u}{\partial x_1} + b_{12} \frac{\partial u}{\partial x_2} + \dots + b_{1n} \frac{\partial u}{\partial x_n} + b_{21} \frac{\partial u}{\partial y_1} + b_{22} \frac{\partial u}{\partial y_2} + \dots + b_{2n} \frac{\partial u}{\partial y_n} + b_{31} \frac{\partial v}{\partial x_1} + b_{32} \frac{\partial v}{\partial x_2} + \dots + b_{3n} \frac{\partial v}{\partial x_n} + \\ &b_{41} \frac{\partial v}{\partial y_1} + b_{42} \frac{\partial v}{\partial y_2} + \dots + b_{4n} \frac{\partial v}{\partial y_n} + d_1 u + d_2 v + d_3 \end{aligned} \quad (1)$$

For two desired real-valued functions $u = (t, x_1 \dots, x_n, y_1 \dots, y_n)$ and $v = (t, x_1 \dots, x_n, y_1 \dots, y_n)$ where t means the time and $(x_1 \dots, x_n, y_1 \dots, y_n)$ runs in a (bounded) domain in the R^n . The coefficients are supposed to depend at least continuously on $t, (x_1 \dots, x_n)$ and $(y_1 \dots, y_n)$. On the one hand, in view of the classical Cauchy- Kovalevskaya theorem the initial value problem

$$u(0, x_1 \dots, x_n, y_1 \dots, y_n) = \varphi(x_1 \dots, x_n, y_1 \dots, y_n)$$

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$$v(0, x_1, \dots, x_n, y_1, \dots, y_n) = \phi(x_1, \dots, x_n, y_1, \dots, y_n)$$

is solvable provided the coefficients of (1) and the initial functions φ, ϕ possess power-series representations corresponding author: Ahmad Neirameh (in (x_1, \dots, x_n) and (y_1, \dots, y_n)).

On the other hand, in view of the famous Lewy example [3] there are systems of the form (1) with infinitely differentiable coefficients such that no solution exists, say nothing of the solvability of the initial value problems (2) and (3). The present article formulates sufficient conditions on the coefficients of (1) under which

each initial value problems (2) and (3) is solvable provided the initial functions φ, ϕ satisfy the Cauchy-

Riemann system. We will see that four coefficients of (1) can be chosen as arbitrary continuous functions. This result will be reached by the technique of associated differential operators applied to a complex rewriting of the initial value problems (1) - (3). A pair F, G of differential operators is said to be associated in case F

transforms solutions u of $Gu = 0$ again into solutions of this equation. An initial value problem of type $\frac{\partial u}{\partial t} = Fu$ possesses solutions belonging to an associated space for each t provided the initial functions

belongs to the associated space and an interior estimate holds in the associated space (see [6]). Sometimes it can happen that an evolution operator F possesses several so-called co-associated spaces.

2- The Complex Form of the Given System:

Usually, we defines the complex variables $z_j = x_j + iy_j, j = 1, 2, \dots, n$ and $w = u + iv$. Then the derivatives of u and v with respect to the real variables x_j and y_j can be expressed by the partial complex derivatives

$$\frac{\partial w}{\partial z_j} = \frac{1}{2} \left(\frac{\partial w}{\partial x_j} - i \frac{\partial w}{\partial y_j} \right)$$

$$\frac{\partial w}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial w}{\partial x_j} + i \frac{\partial w}{\partial y_j} \right)$$

of w with respect to z_j and \bar{z}_j . E.g., one has

$$\frac{\partial u}{\partial x_j} = \frac{1}{2} \left(\frac{\partial w}{\partial z_j} + \frac{\partial \bar{w}}{\partial z_j} + \frac{\partial w}{\partial \bar{z}_j} + \frac{\partial \bar{w}}{\partial \bar{z}_j} \right)$$

$$\frac{\partial v}{\partial x_j} = \frac{1}{2} i \left(\frac{\partial w}{\partial z_j} - \frac{\partial \bar{w}}{\partial z_j} + \frac{\partial w}{\partial \bar{z}_j} - \frac{\partial \bar{w}}{\partial \bar{z}_j} \right)$$

$$\frac{\partial u}{\partial y_j} = \frac{1}{2} i \left(\frac{\partial w}{\partial z_j} - \frac{\partial \bar{w}}{\partial z_j} - \frac{\partial w}{\partial \bar{z}_j} + \frac{\partial \bar{w}}{\partial \bar{z}_j} \right)$$

$$\frac{\partial v}{\partial y_j} = \frac{1}{2} \left(\frac{\partial w}{\partial z_j} + \frac{\partial \bar{w}}{\partial z_j} - \frac{\partial w}{\partial \bar{z}_j} - \frac{\partial \bar{w}}{\partial \bar{z}_j} \right)$$

Then the systems (1) can be rewritten in the form

$$\frac{\partial w}{\partial t} = \frac{1}{2} \sum_{j=1}^n \left[A_j \frac{\partial w}{\partial z_j} + B_j \frac{\partial \bar{w}}{\partial z_j} + C_j \frac{\partial w}{\partial \bar{z}_j} + D_j \frac{\partial \bar{w}}{\partial \bar{z}_j} \right] + Ew + F\bar{w} + G \tag{4}$$

Where

$$A_j = (a_{1j} + b_{2j} - b_{3j} + a_{4j}) + i(b_{1j} - a_{2j} + a_{3j} + b_{4j})$$

$$B_j = (a_{1j} - b_{2j} + b_{3j} + a_{4j}) + i(b_{1j} - a_{2j} - a_{3j} + b_{4j})$$

$$C_j = (a_{1j} - b_{2j} - b_{3j} - a_{4j}) + i(b_{1j} + a_{2j} + a_{3j} - b_{4j})$$

$$D_j = (a_{1j} + b_{2j} + b_{3j} - a_{4j}) + i(b_{1j} - a_{2j} - a_{3j} - b_{4j})$$

$$E = (c_1 + d_2) + i(d_1 - c_2)$$

$$F = (c_1 - d_2) + i(d_1 + c_2)$$

$$G = c_3 + id_3$$

3- The Basic Theorem for the Associated Pairs:

Next we formulate sufficient and necessary conditions under which the operator on the right-hand side of

(4) and the Cauchy-Riemann operator $\frac{\partial w}{\partial \bar{z}_j}$ for man associated pair:

Theorem 3.1 Suppose A_j, B_j, E, F and G are continuously differentiable with respect to z_j and \bar{z}_j

Then the operator on the right-hand side of (4) is associated to the Cauchy-Riemann operator if and only if the following conditions are satisfied:

1. B_j and F are identically equal to zero.
2. A_j, E and G are holomorphic.

Proof: Denoting the right-hand side of (4) by w , and taking into account the Cauchy-Riemann system, one gets

for $\frac{\partial w}{\partial \bar{z}_j}$ the expression

$$\frac{\partial A_j}{\partial \bar{z}_j} \cdot \frac{\partial w}{\partial z_j} + \frac{\partial B_j}{\partial \bar{z}_j} \cdot \frac{\partial \bar{w}}{\partial z_j} + B_j \cdot \frac{\partial \bar{w}}{\partial \bar{z}_j} + \frac{\partial E}{\partial \bar{z}_j} \cdot w + \frac{\partial F}{\partial \bar{z}_j} \cdot \bar{w} + F \cdot \frac{\partial \bar{w}}{\partial z_j} + \frac{\partial G}{\partial \bar{z}_j} \tag{12}$$

Therefore w is holomorphic in case the conditions of the lemma are satisfied.

Now assume, conversely, that w is always holomorphic if only w is so. Choose, especially $W \equiv 0$. Then (12) passes into $\frac{\partial G}{\partial \bar{z}_j}$. Since W is holomorphic as image of the holomorphic function $w \equiv 0$, we conclude

that G is holomorphic. Thus the term $\frac{\partial G}{\partial \bar{z}_j}$ can be omitted in (12). Next we choose $w \equiv i$ and $w \equiv -i$.

For these two choices (12) implies

$$\frac{\partial E}{\partial \bar{z}_j} + \frac{\partial F}{\partial \bar{z}_j} \equiv 0 \quad \text{and} \quad \frac{\partial E}{\partial \bar{z}_j} - \frac{\partial F}{\partial \bar{z}_j} \equiv 0 \quad \text{resp} \quad (13)$$

and, consequently, E and F , too, turn out to be holomorphic necessarily. Similarly $w(z_j) \equiv z_j$ and $w(z_j) \equiv iz_j$ lead to

$$\sum_{j=1}^n \left(\frac{\partial A_j}{\partial \bar{z}_j} + \frac{\partial B_j}{\partial \bar{z}_j} \right) + F \equiv 0 \quad \text{and} \quad \sum_{j=1}^n \left(\frac{\partial A_j}{\partial \bar{z}_j} - \frac{\partial B_j}{\partial \bar{z}_j} \right) - F \equiv 0$$

These two relations show that A_j is holomorphic and, moreover, F and B_j are connected by the relation

$$\sum_{j=1}^n \left(\frac{\partial B_j}{\partial \bar{z}_j} \right) + F \equiv 0$$

Meanwhile we know that A_j, E, F and G are holomorphic. Taking into account relation (13), the expression

(12) now simplifies to $B_j \cdot \frac{\partial w}{\partial z_j}$. It remains to choose $w \equiv z_j^2$. This implies $B_j \equiv 0$. In view of (13)

it follows, finally, that also F vanishes identically. This completes the proof of the theorem 3.1.

4- Description of the Permissible Coefficients:

The above theorem 3.1 determines all complex equations (4) to which the Cauchy-Riemann system is associated. In the sequel we characterize the corresponding real systems (1). In view of (9), (10) and (11), the coefficients c_j and $d_j, j=1,2$ are uniquely determined by the real and imaginary parts of two arbitrary holomorphic functions. Splitting up the Eqs. (5)-(8) into real and imaginary parts, one gets 8 linear equations for the 8 coefficients $a_{ij}, b_{ij}, i=1,2, \dots, n$. The determinant of the coefficients of this linear system is different from zero, i.e., the a_{ij} and b_{ij} are uniquely determined in case the coefficients A_j, B_j, C_j and D_j are given. Observe, however, that C_j and D_j are completely free for systems to which the Cauchy-Riemann system is associated. In other words, for the desired associated systems 4 of the 8 coefficients can be chosen arbitrarily. Notice that A_j must be holomorphic and B_j must vanish identically in view of Theorem 3.1.

5- Initial Value Problem:

In this stage for solving the initial value problems (1)-(3), consider an exhaustion of the (bounded) domain Ω by a family of subdomains Ω_s , $0 \prec s \prec s_0$ satisfying the usual condition $\text{dist}(\Omega_{s_1}, \Omega_{s_2}) > 0$ if only $0 \prec s_1 \prec s_2 \prec s_0$. Let \tilde{h}_s be the Banach space of functions holomorphic in Ω_s and continuous in s . Then the \tilde{h}_s , $0 \prec s \prec s_0$ form a scale of Banach spaces. Rewrite the complex version of the given initial value problem as abstract operator equation in the scale \tilde{h}_s . In view of Cauchy's Integral Formula for the complex derivative of a holomorphic function, the abstract Cauchy-Kovalevskaya theorem(cf. [5 and 6]) is applicable. Consequently, the following theorem has been proved:

Theorem 5.1:

Suppose the coefficients of the systems (1) are given in accordance with Section 4. Suppose, further, that the initial functions φ and ϕ in (2) and (3) satisfy the Cauchy-Riemann system. Then the initial value problems (1)-(3) is solvable. The solution exists at least in the time-interval $0 \leq t < a_*(s_0 - s)$ if a_* is sufficiently small and $(x_1, \dots, x_n, y_1, \dots, y_n)$ belongs to Ω_s (where the subdomains (Ω_s) form an exhaustion of (Ω)).

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