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Homotopy Perturbation Method for the Nonlinear Fractional Integro-differential Equations

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Abstract: In this paper, the homotopy perturbation method is applied to investigate the numerical solutions of the nonlinear fractional integro-differential equations (NFIDEs). The results of applying this method shows the simplicity and effectiveness of it to solve probable solutions of NFIDE.

Key words: Homotopy perturbation method; Nonlinear integro-differential equations; Fractional differential equations

INTRODUCTION

Mathematical modeling of real-life problem usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. Many mathematical formulations of physical phenomena contain integro-differential equations. These equations arise in fluid dynamics, biological models and chemical kinetics (Golberg, 1979; Jerri, 1971). Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution.

The homotopy perturbation method (HPM) (Abbasbandy, S., 2006; 2007; An and Chen, 2008; Ghasemi *et al.,* 2007; He 1998; 1999; 2003; 2004a; 2004b; 2005; Waz waz, 1997) is a general analytic approach to get series solutions of various types of nonlinear equations. The HPM is based on Homotopy, a fundamental concept in topology and differential geometry (Sen, 1983).

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential and integro-differential equations of fractional order (Caputo, 1967; Ganjiani, 2010; Momani and Odibat., 2007; Odibat and Momani, 2008; Podlubuy, 1999; Tarasov, 2009; Wang, 2007; West *et al.,* 2003). Some existence and uniqueness results in a Banach space for a fractional integro-differential equations are introduced by Ahmad and Sivasundaram (Ahmad and Sivasundaram. 2008).

In this article, we would like to apply the HPM to the solution NFIDE of the form:

$$D^{\alpha}u(x) = g(x) + \int_0^x k(t, u(t), D^{\alpha}u(t))\,dt, \qquad x > 0,\ 0 < \alpha \le 1, \tag{1}$$

Where α is a parameter describing the order of the fractional derivative. The function *u(x)* is unknown and assumed to be causal i.e. u(x)=0, for x<0 and for and continuous functions *g, k* are known. Comparison are made between the exact solution, Wavelet-Galerkin method (Ghasemi *et al.,* 2007) and HPM.

2. Preliminaries and Notations:

In this section we want to review some of the basic definitions of functional calculus, which have been given in books (Podlubuy, 1999; West *et al.,* 2003). There are various definitions of fractional integration and differentiation, such as Riemann-Liouville's definition, Caputo's definition and generalized function approach. For the purpose of this study the Caputo's definition of fractional differentiation will be used.

Definition 2.1:

Caputo's definition of the fractional-order derivative is defined as

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$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \, d\tau, \qquad (n-1 < Re(\alpha) \le n, \qquad n \in N), \tag{2}$$

where the parameter α is the order of the derivative and α is allowed to be real or even complex α, is the initial value of function *f*. In this paper only real and positive will be considered. For the Caputo's derivative we have

$$D^{\alpha} c = 0, \qquad (c \text{ is a constant})$$

$$D^{\alpha} t^{\beta} = \begin{cases} 0, & (\beta \le \alpha - 1), \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\beta > \alpha - 1). \end{cases}$$

Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$D^{\alpha}\left(\lambda f(t) + \mu g(t)\right) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} g(t),$$

Where λ, μ are constants and, satisfies the so-called the Leibnitz rule:

$$D^{\alpha}(g(t) f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) \, D^{\alpha-k} f(t),$$

If *f(r)* is continuous in [a,t] and g(r) has n+1 continuous derivatives in [a,t]. For establishing our results, we also necessarily introduce following Riemann-Liouville fractional integral operator.

Definition 2.2:

A real function $f(x)$, $x>0$, is said to be in the space C_{μ}, $\mu \in R$ if there exist a real number $p(>\mu)$ such that, $f(x)=x^{p} f_1(x)$ where $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_{μ}^{m} iff $f(m) \in C_{\mu}$, $m \in N$.

Definition 2.3:

The Riemann-Liouville fractional integral operator of order $a \ge 0$, of a function $f \in C_{\mu}$, $\mu \ge -1$, is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t) \, dt, \quad \alpha > 0, \qquad x > 0. \tag{3}$$

Properties of the operator can be found in Ref. (West *et al.,* 2003) and we mention only some of them in the following : For $f \in C_{\mu}$, $\mu \ge -1$, α, $\beta \ge 0$, $Y > -1$:

$$J^{0} f(x) = f(x), \qquad J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma},$$

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x), \qquad J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x).$$

Also, here we need two of its basic properties. If $m-1 < a \le m$, $m \in N$ and $f \in C^{m}_{\mu}$, $\mu \ge -1$ then

$$D^{\alpha} J^{\alpha} f(x) = f(x), \qquad J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{i=0}^{m-1} f^{i}(0^{+}) \frac{x^{i}}{i!}, \qquad x > 0.$$

3. Homotopy Perturbation Method and Applications:

Let us consider the NFIDE

$$D^{\alpha}u(x) = g(x) + \int_0^x k(t, u(t), D^{\alpha}u(t))dt, \quad x > 0, \quad 0 < \alpha \le 1, \tag{5}$$

According to the homotopy perturbation method , we construct the following simple homotopy:

$$(1-p)\,(D^{\alpha}u(x) - g(x)) + p(\,D^{\alpha}u(x) - g(x) - \int_0^x k(t, \quad u(t), \quad D^{\alpha}u(t))dt) = 0,$$

or

$$D^{\alpha}u(x) - g(x) - p\,(\int_0^x k(t, u(t), D^{\alpha}u(t))dt\,) = 0, \tag{7}$$

where p $\in$[0,1] is an embedding parameter. The HPM uses the homotopy parameter as an expanding parameter (Nayfeh, 1985) to obtain

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots, \tag{8}$$

and

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \tag{9}$$

The series (9) is convergent for most cases (He, 2003).

4 Applications:

In order to illustrate the advantages and the accuracy of the HPM to solving nonlinear fractional integro-differential equations , we have applied the method to two examples.

Example 1:

Consider the nonlinear fractional integro- differential equation

$$D^{\alpha}u(x) = 1 + \int_0^x u(t)D^{\alpha}u(t)\,dt, \quad 0 \le x \le 1,\ 0 \le \alpha \le 1. \tag{10}$$

For α = 1 the exact solution is $u(x) = \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}x\right).$ We may choose a convex homotopy by

$$H(u\,;p) = D^{\alpha}u(x) - g(x) - p\int_0^x k(t, u(t), D^{\alpha}u(t))dt = 0. \tag{11}$$

Substituting (8) into (11) and equating the terms with identical powers of ρ, we have

$$p^0:\ D^{\alpha}v_0(x) = g(x) \ \Rightarrow\ v_0(x) = c_0x^{\alpha}, \quad \text{where} \quad c_0 = \frac{1}{\Gamma(\alpha+1)},$$

$$p^1:\ D^{\alpha}v_1(x) = \int_0^x v_0(t)D^{\alpha}v_0(t)\,dt \ \Rightarrow\ v_1(x) = c_1x^{2\alpha+1}, \quad \text{where} \quad c_1 = c_0\,\frac{\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+2)},$$

$$p^2:\ D^{\alpha}v_2(x) = \int_0^x (\,v_0(t)D^{\alpha}\,v_1(t) + v_1(t)D^{\alpha}v_0(t)\,)\,dt \ \Rightarrow\ v_2(x) = c_2x^{3\alpha+2},$$

where $c_2 = \frac{(c_o^2/(a+1)+c_1)}{(2a+2)/\Gamma(3a+3)}$

$$p^3:\ D^{\alpha}v_3(x) = \int_0^x (v_0(t)D^{\alpha}v_2(t) + v_1(t)D^{\alpha}v_1(t) + v_2(t)D^{\alpha}v_0(t))\,dt \ \Rightarrow\ v_3(x) = c_3x^{4\alpha+3},$$

where $$c_3 = \frac{\Gamma(3\alpha+4)}{\Gamma(4\alpha+4)\left(\frac{c_0c_2\Gamma(3\alpha+3)}{\Gamma(2\alpha+3)} + \frac{c_0c_1}{\alpha+1} + c_2\right)},$$

and so on.

The approximation solution of (10) in finite series form is given by

$$u \simeq v_0 + v_1 + v_2 + v_3 + \cdots. \tag{12}$$

The evaluation results for the exact solution (10), for the special case α = 1 and the approximate solutions (10), with six terms and different values of α, are shown in Fig. 1. The error of approximate solution for α = 1 is shown in Fig. 2. It can be seen from Fig .1 and Fig. 2. that the solution obtained by the presented method is nearly identical with the exact solution. Then, we may conclude that we have achieved a good approximation with the exact solution of the equation by using the first few terms only. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series (9).

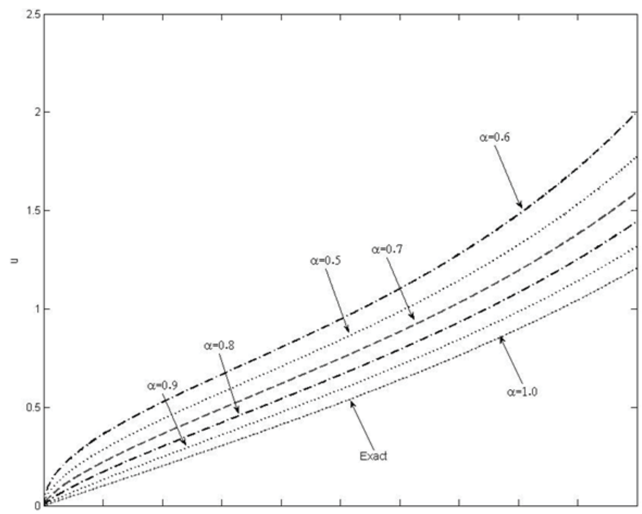

Fig. 1: The exact solution for (10) and approximate solutions with six terms.

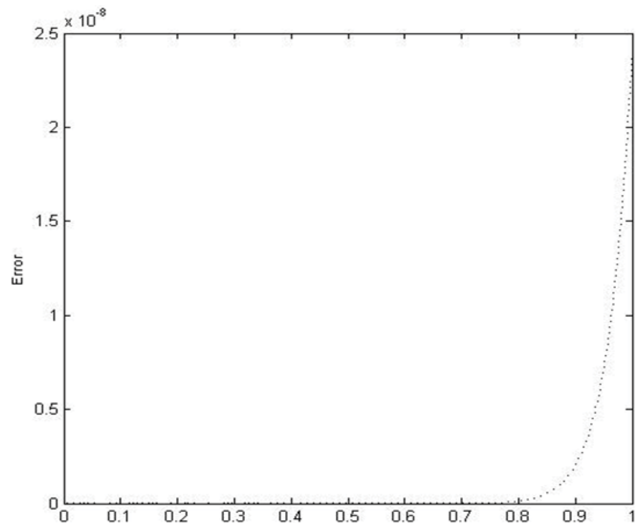

Fig. 2: The error of approximate solution for α = 1.

Example 2:

Consider

$$D^{\alpha}u(x) = -1 + \int_0^x u(t)^2 dt, \qquad 0 \le x \le 1, \qquad 0 < \alpha \le 1, \tag{13}$$

with the boundary solution $u(0)=0$ we find

$$p^0: D^{\alpha}v_0(x) = g(x) \implies v_0(x) = c_0 x^{\alpha}, \qquad \text{where} \quad c_0 = \frac{-1}{\Gamma(\alpha+1)},$$

and so on

Fig. 3 shows the approximate solutions with six terms and different values of α. The error of approximate solution for α = 1 is shown in Fig. 4. Table.1 contains a numerical comparison between our solutions using HPM for different values of α and the solutions of the same problem presented in (Ghasemi *et al.,* 2007) using Wavelet-Galerkin method (WGM) and homotopy perturbation method.

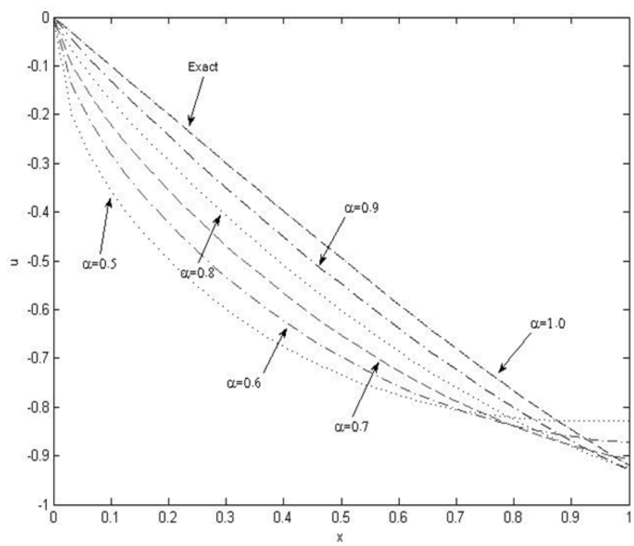

Fig. 3: The exact solution for (13) and approximate solutions with six terms.

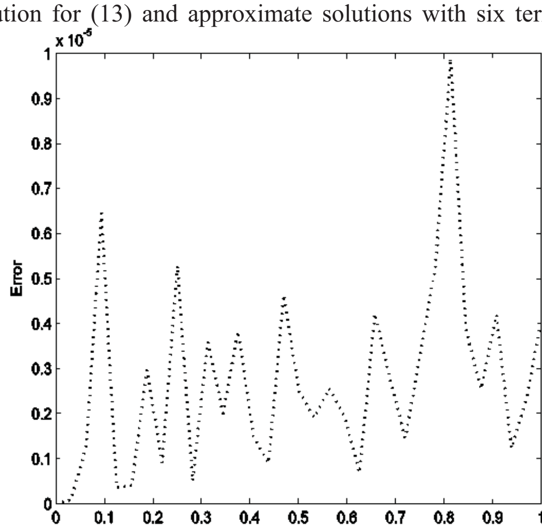

Fig. 4: The error of approximate solutions for α = 1.

Table 1: Numerical results of Example 2.

x	Exact solution	WGM	HPM(α=1)	HPM(α=0.9)	HPM(α=0.8)
0	0	0	0	0	0
0.0312	0.03120-	0.0312-	0.03120-	0.04588-	0.06702-
0.0625	0.06250-	0.0625-	0.06250-	0.08574-	0.11682-
0.0938	0.09380-	0.0937-	0.09380-	0.12355-	0.16162-
0.125	0.12498-	0.1250-	0.12498-	0.15997-	0.20328-
0.1562	0.15615-	0.1562-	0.15615-	0.19542-	0.24282-
0.1875	0.18740-	0.1874-	0.18740-	0.23024-	0.28082-
0.2188	0.21861-	0.2186-	0.21861-	0.26441-	0.31742-
0.25	0.24968-	0.24970-	0.24968-	0.29790-	0.35272-
0.2812	0.28068-	0.28070-	0.28068-	0.33056-	0.38695-
0.3125	0.31171-	0.3117-	0.31171-	0.36342-	0.42039-
0.3438	0.34264-	0.3426-	0.34264-	0.39549-	0.42527-
0.375	0.37336-	0.3734-	0.37336-	0.42689-	0.48413-
0.4062	0.40394-	0.4040-	0.40394-	0.45799-	0.51470-
0.4375	0.43446-	0.4345-	0.43446-	0.48861-	0.54451-
0.4688	0.46479-	0.46479-	0.46479-	0.51874-	0.57347-
0.5	0.49482-	0.4948-	0.49482-	0.54824-	0.60150-
0.5312	0.52461-	0.52461-	0.52461-	0.57721-	0.62869-
0.5625	0.55423-	0.5542-	0.55423-	0.60571-	0.65510-
0.5938	0.58354-	0.5835-	0.58354-	0.63363-	0.68063-
0.625	0.61243-	0.6124-	0.61243-	0.66086-	0.70521-
0.6562	0.64095-	0.6410-	0.64095-	0.68744-	0.72888-
0.6875	0.66917-	0.6692-	0.66917-	0.71345-	0.75172-
0.7188	0.69694-	0.6969-	0.69694-	0.73876-	0.77363-
0.75	0.72415-	0.7242-	0.72415-	0.76327-	0.79451-
0.7812	0.75086-	0.7509-	0.75085-	0.78702-	0.81442-
0.8125	0.77710-	0.7771-	0.77710-	0.81006-	0.83341-
0.8438	0.80273-	0.8027-	0.80273-	0.83228-	0.85140-
0.875	0.82767-	0.8277-	0.82767-	0.85359-	0.86831-
0.9062	0.85193-	0.8520-	0.85193-	0.87402-	0.88419-
0.9375	0.87557-	0.8756-	0.87557-	0.89361-	0.89908-
0.9688	0.89845-	0.8984-	0.8984-	0.91225-	0.91291-
1	0.92048-	0.9205	0.92048-	0.92988-	0.92562-

5. Conclusion:

In this paper, the homotopy perturbation method has been applied for the numerical solutions of the nonlinear fractional integro-differential equations. The results obtained in this work confirm the notion that the HPM is a powerful and efficient technique for finding numerical solutions for these kind of equations.

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