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Numerical Solution of Singular Integral Equations Using Haar wavelet

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Abestract: A new computational method for solving Abel's integral equation as a singular Volterra integral equation is presented. The method is based on Haar wavelets approximation. Numerical examples show the validity and applicability of the method.

Key words: Abel's integral equation, Haar wavelets, Singular Volterra integral equation.

INTRODUCTION

Abel's integral equation is one of the most important integral equations that derived directly from a concrete problem of mechanics or physics (without passing through a differential equation) (Atkinson, 1997; Baker, 1977; Wazwaz, 1997).

Historically, Abel's problem is the first one to lead to the study of integral equations. In 1823, Abel, when generalizing the tautochrone problem derived the equation

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt = f(x),$$

where f(x) is a known function and y(x) is an unknown function to be determined. This equation is a particular case of a linear Volterra integral equation of the first kind. The generalized Abel's integral equations on a finite segment appeared for the first time in the paper of Zeilon, (1924).

Several numerical methods for approximating the solution of singular integral equations are known. Baker Zeilon, (1924) studied the numerical treatment of integral equations. A numerical solution of weakly singular Volterra integral equations was introduced in (Baratella and Orsi, 2004).

Babolian and Salimi (Babolian and Salimi Shamloo, 2008) discussed a operational matrix method based on block-pulse functions for singular integral equations. In (Yousefi, 2006), Legendre wavelets approximation have been used for numerically solution of Abel's integral equations.

One of the most attractive proposals made in the last years was an idea connected to the application of wavelets as basis functions in the numerical solution of integral equations. The wavelets technique allows the creation of very fast algorithms when compared to the algorithms ordinarily used.This is due to specific attributes when it is used as a basis function. Various wavelet basis are applied. In addition to the conventional Duabechies wavelets linear B-spline wavelets (Lewis, 1973) have been used.

This paper presents the application of Taar wavelets as a basis functions for numerical solution of the Abel's integral equation of the form

$$\lambda\, y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \qquad 0 \le x, t \le 1 \tag{1}$$

Where $\lambda = 0 \text{ or } \lambda = 1$

Our method consists of applying Haar wavelets to solve equation (1) and reduce it to a set of algebraic equations by expending the unknown function as these wavelets series with unknown coefficients. The properties of these basis functions are then utilized to evaluate the unknown coefficients.

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Definition:

The Haar wavelet is function defined on real line as:

$$H(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & elsewhere \end{cases}$$

Now for n =1, 2 ,…, write $n=2^j$ with j=0,1,…, 2^j-1 and define $h_n(t) = 2^{\frac{j}{2}} H(2^j t - k)|_{[0,1]}$.

Also, define $h_0(t)=1$ for all t. Here the integer $2^j, j$ =0,1,..., indicates the level of the wavelet and k=0,1,…, 2^j-1 is the translation parameter. It can be shown that the sequence $\{h_n\}_{n=0}^{\infty}$ is a complete orthonorma system in $L^2[0,1]$ and for $f \in c[0,1]$ the series $\sum_n \langle f, h_n \rangle h_n$ converges to uniformly to f (Wojtaszez, 1997), where $\langle f, h_n \rangle = \int_0^1 f(x) h_n(x) dx.$

3.Function approximation:

A function $u(t)$ defined over interval [0,1) may be expanded as:

$$u(t) = \sum_{n=0}^{\infty} u_n h_n(t), \tag{2}$$

with $u_n = \langle u(t), h_n(t) \rangle$,

In practice, only the first k-term of (2),that is consideres,where k is a power of 2,that is,

$$u(t) \simeq u_k(t) = \sum_{n=0}^{k-1} u_n h_n(t), \tag{3}$$

with matrix form:

$$u(t) \simeq u_k(t) = \boldsymbol{U}^T \boldsymbol{H}(t) \tag{4}$$

Where ,

$$\boldsymbol{U}(t) = [u_0, u_1, \dots, u_{k-1}]^T \quad , \quad \boldsymbol{H}(t) = [h_0(t), h_1(t), \dots, h_{k-1}(t)]^T \tag{5}$$

4.Method of Solution:

In this section we solve singular Volterra integral equation (1) by using Haar wavelet .
We now approximate $y(x)$ and $f(x)$ by the way mentioned in section 3 as

$$y(x) = \boldsymbol{Y}^T \boldsymbol{H}(x), \qquad f(x) = \boldsymbol{F}^T \boldsymbol{H}(x) \tag{6}$$

Where $H(x)$ is defined in (5),Y is k ×1 unknown vector defined similary to U in (5) and coefficients of F are known.
From Eqs. (1) and (6) we get

$$\lambda \boldsymbol{Y}^T \boldsymbol{H}(x) = \boldsymbol{F}^T \boldsymbol{H}(x) + \int_0^x \frac{\boldsymbol{Y}^T \boldsymbol{H}(t)}{\sqrt{x-t}} dt. \tag{7}$$

we get

$$\int_0^x \frac{\boldsymbol{H}(t)}{\sqrt{x-t}}\,dt = \boldsymbol{\Lambda}\,\boldsymbol{H}(x) \tag{8}$$

where A is a k×*k* matrix. Then from Eqs. (7) and (8) we have

$$\lambda \boldsymbol{Y}^T \boldsymbol{H}(x) = \boldsymbol{F}^T \boldsymbol{H}(x) + \boldsymbol{Y}^T \boldsymbol{\Lambda}\,\boldsymbol{H}(x) \tag{9}$$

or

$$\lambda \boldsymbol{Y}^T = \boldsymbol{F}^T + \boldsymbol{Y}^T \boldsymbol{\Lambda} \tag{10}$$

The resulting equation (10) generates a system of k linear equations which can be solved using direct or iterative methods and the solution is

$\boldsymbol{Y}^T = -\boldsymbol{F}^T.\boldsymbol{\Lambda}^{-1}$ for λ=0

or

$\boldsymbol{Y}^T = \boldsymbol{F}^T.(\boldsymbol{I}-\boldsymbol{\Lambda})^{-1}$ for λ=0

Thus $y(x)=Y^T H(x)$ is the solution of (1).

5.Illustrative Examples:

In this section, we applied the method presented in this paper for solving integral equation (1) and solved some examples The computations associated with the examples were performed using Matlab.

Example 1:

Consider the following Abel's integral equation of the first kind

$$x = \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt,$$

Table 1: Numerical results for Example 1

x_i	Exact solution	Approximate solution K= 4
0.0	0.000000	0.105133
0.1	0.201317	0.191236
0.2	0.284705	0.270428
0.3	0.348691	0.379425
0.4	0.402634	0.383214
0.5	0.450158	0.447158
0.6	0.493124	0.473466
0.7	0.532634	0.561730
0.8	0.569410	0.590881
0.9	0.603951	0.664391

with exact solution The numerical results represented in Table 1.

Example 2:

As the second example consider the following Abel's integral equation of the first kind with exact solution Table 2 illustrates the numerical results for Example 2.

6.Main Results:

in this work, we solved Abel's integral equations of the first and second kind by using Haar wavelet.

Haar wavelet are well behaved basic functions that are orthonormal on [0, 1]. We can modify this method for the numerical solution of nonlinear singular integral equations of the first and second kind.

Table 2: Numerical results for Example 2

x_i	Exact solution	Approximate solution K= 4
0.0	1.000	0.0123
0.1	0.991	0.9524
0.2	0.968	0.9451
0.3	0.937	0.9253
0.4	0.904	0.8921
0.5	0.875	0.8461
0.6	0.856	0.8423
0.7	0.853	0.8715
0.8	0.872	0.8843
0.9	0.919	0.9621

REFERENCES

Atkinson, K.E., 1997. The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, Cambridge.

Baker, C.T.H., 1977. The Numerical Treatment of Integral Equations, Clarendon Press, Oxford.

Baratella, P., A.P. Orsi, 2004. ”A new approach to the numerical solution of weakly singular Volterra integral equations,” J. Comp. Appl. Math., 163: 401-418.

Babolian, E., A. Salimi Shamloo, 2008. ”Numerical solution of Volterra integral and integro-differential equations of convolution type by using operational matrices

Lewis, B.A., 1973. ”On the numerical solution of Fredholm integral equations of the first kind,” J. Inst. Math. Appl., 16: 207-220.

Wazwaz, A.M., 1997. A First Course in Integral Equations, World scientific Publishing Company, New Jersey.

Wojtaszczyk, P., 1997. A Mathematical Introduction to Wavelets, Cambridge University Press.

Yousefi, S.A., 2006. ”Numerical solution of Abel’s integral equation by using Legendre wavelets”, Appl. Math. Comp., 175: 574-580.

Zeilon, N., 1924. ”Sur quelques points de la theorie de l’equation integrale d’Abel,”Arkiv. Mat. Astr. Fysik., 18: 1-19.