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On Some Fundamental Concepts Governing Continuum Models of Porous Ductile Solids

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Abstract: This paper present a survey of some of the concepts concerning continuum models of porous ductile solids. The paper survey the work of Viggo Tvergaard (1990), Andersson (1977) and Berg (1970). It is a well known fact that failure by coalescence of microscopic voids is an important fracture mechanism in ductile metals. The paper will revisit some of the fundamental concepts governing material failure by void growth to coalescence.

Key words: Void, Material failure, Coalescence. Ductile

INTRODUCTION

It is a known fact that failure by coalescence of microscopic voids is an important fracture mechanisim in ductile metals. It is also a documental fact that voids nucleate mainly at second phase particles, decohesion of the particle-matrix interface or by particle fracture. The voids growth due to plastic straining of the surrounding material (c.f. Tuergaard 1990).

Tuergaad (1990) reported that the first micromechanical studies of the above phenomena was reported by McClintock 1968, Rice and Tracy 1969, and it focused on the growth of a single void of an infinite elastic-plastic solid of which the results were used to estimate critical strains for coalescence. There had been many research works on porous ductile material models. The most widely known porous ductile material model is that one developed by Grurson (1977a). This was based on averaging techniques which was similar to those developed by Bishop and Hill (1951). (c.f. Tvergaard 1990)

Grurson (1977a) considered the characteristic volume element as an aggregate of voids and rigid plastic matrix material rather than a polycrystalline aggregate, and approximate upper-bound solutions on the micro-level and this had been used to derive a macroscopic yield condition for the material (c.f. Tvergaard 1990).

This survey is divided into four sections. Section two will discuss the basic equations. Section three will survey the continuum Models of porous Ductile solids in relationship with among other things; Approximation yield conditions, Nucleation and growth of voids and stress – strain relations.

The last section will survey the work of Feucht Gress and Wriggers (1997) as regard the application of Tvergaad (1990) to Damage Mechanics, more importantly, Element, formulation for Nonlocal Damage Models.

The Fundamental Equations:

The growth of voids to coalescence by plastic yielding of the surrounding material involves many geometrical changes which make it necessary to involve finite strain formulations of the field equations. This also applies to micromechanical studies which focus on the local stress and strain fields around each single void as well as to studies based on continuum models that describe the average macroscopic effect of porosities in ductile solids (c.f. Tvergaard (1990).

We shall make the following definitions.

The vector r will represent the position of a material point in the reference configuration. It must be stated here that the finite strain formulation of Tvergaard 1990 which is based on a Langrangian framework, in which the reference configuration is usually identified with the initial underformed configuration is adopted in this review.

$\bar{r}$ is the position of the same point in the current configuration

u is the displacement while F is the deformation gradient.

The relationship among the above defined parameters can be put in the form of;

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$$u = \bar{r} - r,\ F = \frac{\partial \bar{r}}{\partial r} \qquad (1)$$

By using Green and Zerna (1968), and Budiansky 1969 as regards the converted co-ordinate formulation of the governing equations we can introduce converted co-ordinate μ^i which serve as what in refered to as particle labels.

We now consider the displacement vector u as a function of the co-ordinates μ^i and a monotonically increasing time like parameter t ; i = 1,2,3.

We now define what is refered to as the covariant base vectors e_i and e_i of the material net in the reference configuration and the current configuration as;

$$e_i = \frac{\partial r}{\partial \mu^i}\ ,\quad \bar{e}_i = \frac{\partial \bar{r}}{\partial \mu^i} \qquad (2)$$

The matrix tensors in the reference and current configuration g_{ij} and G_{ij} are given by the dot products of the base vectors

$$g_{ij} = e_i.e_j\ ,\quad G_{ij} = \bar{e}_i\ .\ \bar{e}_j \qquad (3)$$

We denote the determinant of g_{ij} and G_{ij} as g and G respectively. We also denote the inverse of the above two matrix as g^{ij} and G^{ij}

Adopting the argument of Tvergaard (1990), we can obtain the components of vectors and tensors on the embedded co-ordinates by dot products with the appropriate base vectors. Hence we obtain the following equations

$$u_i = e_i\ .\ u,\ u^i = e^i.\ u\ ,\ u = u^i\ e_i \qquad (4)$$

If we substitute equation (4) in equation (1) and we apply the result in equation (2), we obtain the following equation

$$\bar{e}_i = e_i + u,_i^k\ e_k \qquad (5)$$

where $u,_i^k$ is the covariant derivative in the reference frame

We can use equation (3) and (5) to obtain the Langrangian strain tensor

$\varepsilon_{ij} = \frac{1}{2}(G_{ij} - g_{ij})$ interns of displacement components as

$$\varepsilon_{ij} = \frac{1}{2}\ (u_{i,j} + u_{j,i} + u^k_{,i}\ u_{k,j} \qquad (6)$$

The true stress tensor s in the current configuration which is the Cauchy stress tensor has the contravariant components s^{ij} on the current base vectors. Hence we have

$$\sigma = \sigma^{ij}\bar{e}_i\ \bar{e}_j,\ \sigma^{ij} = \bar{e}^i \cdot \sigma \cdot \bar{e}^j \qquad (7)$$

We shall define the contravariant component t^{ij} of the Kirchhoff stress tensor on the current base vectors as;

$$\tau^{ij}\sqrt{\frac{G}{g}}\sigma^{ij} \qquad (8)$$

where $\sqrt{\frac{G}{g}} = \frac{d\hat{v}}{dv} = \frac{\rho}{\hat{\rho}}$ which is expressed interms of the volume element dv and the density ρ.

The condition for equilibrium require that

$$\int_v t^{ij}\ \delta_{ij}\ dv = \int_s T^i \delta u_i ds \qquad (9)$$

where v and s are the volume and surface respectively, of the body in the reference configuration. The surface tractions T = T^{ij} e_i per unit area in the reference configuration have the components on the reference base vector

$$T^{ij} = (t^{ij} + t^{kj}\ u^i, k)n_j \qquad (10)$$

with n = n_j e^j representing the surface normal in the reference state (c.f. Tvergaar 1990)

If we assume that the following field quantities are known; stress t^{ij}, displacement u_i and traction T^i, then equation (9) can be expanded to give the following equation;

$$\Delta t\tau \int_v (^{ij}\ \delta\varepsilon_{ij} + \tau^{ij}\ \dot{u}^k_{,i}\ \delta u_{k,j})dv$$

$$= \Delta t\tau \int_s \dot{T}^i\ \delta u_j\ du_i\ ds - [\int_s \tau^{ij}\ \delta\varepsilon_{ij}\ dv - \int_s T^i\ \delta u_i\ ds\ ] \qquad (11)$$

In the constitutive relations, we shall assume that the total strain rate is equal to the sum of the elastic and plastic parts, i.e $\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}$. Thus with an elastic relationship of the form $\overset{\Delta}{\sigma}{}^{ij} = R^{ijkl}\ \sigma\Delta^{ij}\ \varepsilon^e_{kl}$, the consititutive relations can be written in the form;

$$\overset{\nabla}{\sigma}{}^{ij} = R^{ijkl}\ ((\varepsilon_{kl}, -\varepsilon^p_{kl} \qquad (12)$$

where $\overset{\nabla}{\sigma}{}^{ij}$ is the Jaumann (co-rotational) rate of the Cauchy stress tensor. This is related to the converted rate by

$$\overset{\nabla}{\sigma}{}^{ij} = \sigma^{ij} + (G^{ik}\ \sigma^{il} + G^{jk}\sigma^{il})\ \varepsilon_{kl} \qquad (13)$$

The finite strain generalization R^{ijkl} can be given as

$$R^{ijkl} = \frac{E}{1+v}\left\{\frac{1}{2}(G^{ik}G^{ji} + G^{il}G^{ik}) + \frac{v}{1-2v}G^{ij}G^{kl}\right\} \qquad (14)$$

E = Young's modulus and v is Poisson's ratio.

Comment:

As reported in Tvergaad 1990, the elastic increment stress-strain relationship with moduli specified in (14) is reported to be hypo-elastic. This is because it cannot be derived from a work potential, However, it is a known fact that in the limit of small stress relative to Young's modulus the relationship reduces to Hooke's law on the current base vectors. In the elastic-plastic analyses, the elastic contribution to the total straining is usually very small so that the use of hypo-elastic relationship rather than a truly elastic one is a reasonable approximation (c.f. Tvergaard 1990).

For the above reason we need an incremental stress-strain relationship $\dot{\tau}^{ij}$ in equation (11). Using equation (8) (12) and (13), we obtain

$$\tau^{ij} = \sqrt{\frac{G}{g}}\dot{\sigma}^{ij} + \tau^{ij}\ G^{kl}\ \varepsilon_{kl} \qquad (15)$$

In the case of time-independent plasticity, where t is a loading parameter, the resulting constitutive relations can be written as

$$\dot{\tau}^{ij} = L^{ijkl.}\ ^{s}{}_{kl} \tag{16}$$

Viscous material models are known to be time-dependent, so that t denotes the time. In visco-plasticity, where $\dot{l}^{p}_{ij}$ represents non-linearly viscous behaviour, the constitutive relations are of the form

$$\dot{\tau}^{ij} = L_*^{ijkl}\ ^{s}{}_{kl} + \dot{\tau}^{ij}_{*} \tag{17}$$

$\dot{\tau}^{ij}_{*}$ represent part of the viscous terms.

We shall now proceed to discuss some of the concepts governing continuum models of porous ductile solids. These concepts include approximate yield conditions, nucleation and growth of voids and stress-strain relations

Some Fundermental Concepts Governing Continuum Models of Porous Ductile Solids:

We shall discuss in this section, three main concepts namely;

(a) The approximate yield conditions
(b) Nucleation and growth of voids and
(c) The stress-strain relations.

Approximate yield conditions:

(a) It is a well documented fact that the micromechanical studies of void growth in ductile materials was initially focused on the behaviour of a single void in an infinite block of plastic material. Gurson (1977a,b) suggested the use of an approximate yield condition of the form

$\Phi(\sigma^{ij}, \sigma_m, f) = 0$ for a certain volume fraction f of voids. We define σ^{ij} as the average macroscopic Cauchy stress tensor, and σ_m is an equivalent tensile flow stress representing the actual microscopic stress rate in the matrix material.

Gurson (1977a) analysed a number of rigid-plastic upper bound to evaluate the yield function of the form $\Phi = 0$. Some of these analyses focused on a cylindrieal volume element, containing a concentric cylindrical void, some focused on the analogous spherical growth and in some of the cases the effect of elastic unloading regions around the void was equally investigated (c.f. Tvergaard 1990).

We shall represent the approximate yield condition derived on th basis of a spherical model problem with no unloading by;

$$\Phi = \frac{\sigma_e^2}{\sigma_m^2} + 2q_1 f^* \cosh\left(\frac{q_2}{2}\frac{\sigma_k^k}{\sigma_m}\right) - \left[1 + \left(q_1 f^*\right)^2\right] = 0 \tag{18}$$

where $q_1 = q_2 = 1$

$f^* = f$. $\sigma_e = \left(3s_{ij}\, s^{ij}/2\right)^{1/2}$ is the macroscopic Mises stress and $S_{ij} = \sigma^{ij} - G^{ij}\ \sigma_k^k\ /3$ is referred to as the stress deviator. With f = 0, equation (18) reduces to the standard Mises yield condition

Tvergaard (1981, 1982a) carried of micromechanical studies for materials containing period distributions of cylindrical voids or spherical voids and compared with predictions based on the yield condition specified in (18). With $q_1 = 1.5$ and $q_2 = 1$, it was discovered that there was an improvement in the approximation obtained using (18)

Final failure of a porous ductile solid occurs by void coalescence. In Gurson model, loss of material stress carrying capacity occurs when the voids have grown so large that equation (18) has shrunk and this occur when $f = 1/q_1$. A micromechanical model study by Andersson (1977) for growth of a spherical void in the stress field in front of a crack tip give coalescence at f = 0.25.

Tvergaard (1982c) discussed some ways of incorporating the effect of coalescence into the Gurson model. Tvergaard and Needleman (1984) introduced the function f*(f) to model the complete loss of material stress carrying capacity at a realistic void volume. This function is represented by

$$f^*(f)\begin{cases} f & for\ f \le f_e \\ f_c - \dfrac{f_u^* - f_c}{f_F - f_c}\ f - f_c & for\ f > f_c \end{cases} \quad (19)$$

f_F = void volume of final fracture. $F^*(f_F) = f^*_U \ = \dfrac{1}{q_1}$ Shima and Oyene (1976) proposed yield function of the form;

$$\Phi_s = \frac{\sigma_e^2}{\sigma_m^2} + 6.20 f^{1.028} \frac{(\sigma_k^k/3)^2}{\sigma_m^2} - (1 - f)^5 = 0 \quad (20)$$

The yield function (20) was fitted with experiments for sintered copper. Tvergaar (1990) specified that with adoption of (20), a reasonable agreement was obtained with some experimental results for sintered iron and sintered aluminum.

(b) Nucleation and growth of voids:

Tvergaard (1990) indicated that in the process of plastic yielding, the void volume fraction changes. This is partly due to the void growth and to the nucleation of new void. The rate of growth of the void volume can be written in the form.

$$\dot{f} = (\dot{f})growth + (\dot{f})\ \text{nucleation} \quad (21)$$

but

$$(\dot{f})\ \text{growth} = (1 - f)\ G^{ij}\ \dot{\varepsilon}_{ij}^{p} \quad (22)$$

where $\dot{\varepsilon}_{ij}^{p}$ is the plastic part of the macroscopic strain increment. Gurson (1977b) discussed various nucleation models. Needleman and Rice (1978) proposed the following model.

$$(\dot{f})\ \text{nucleation} = A\ \dot{\sigma}_m + B\ \left(\sigma_k^k\right)^{\bullet}/3 \quad (23)$$

The effective plastic strain $\in_m^p$, representing the microscopic strain-state in the matrix material is related to s_m by the following increment equation,

$$\dot{\in}_m^p = \left(\frac{1}{E_t} - \frac{1}{E}\right)\dot{\sigma}_m \quad (24)$$

where E_t is the slope of the uniaxial true stress-natural strain curve for matrix material at the current stress level s_m It is now clear that a material for which void nucleation is controlled by the plastic strain can be modelled using (23) if we take A > 0 and B = 0. As reported in Chu and Needleman (1980), the value of A and B can be given as

$$A = \left(\frac{1}{E_t} - \frac{1}{E}\right)\frac{f_N}{s\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\in_m^p - \in_m}{s}\right)^2\right] \quad (25)$$

and

$B = 0$ for $\in_m^p = (\ \in_m^p\ )_{max}$ and $\dot{\in}_m^p > 0$

We define $\in_N$ as the mean strain for nucleation, s is the corresponding standard deviation while f_N is the volume fraction of void nucleation particles (c.f. Tvergaard 1990)

Needleman and Rice (1978) had another suggestion for material in which nucleation is controlled by maximum stress on the particle-matrix interface. They suggested using the sum σ_m + $/\sigma_k^k$ as an approximate measure of minimum stress, by taking A = B. This produces the value of A and B as,

$$A = B = \frac{fN}{s\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{(\sigma_m + \sigma_k^k\ /3)\text{-}\sigma_N}{s}\right]^2\right\} \tag{26}$$

for σ_m + σ_k^k /2 = ($\sigma_{m\,+}$ σ_k^k) $_{max}$, ($\sigma_{m\,+}$ σ_k^k /3)$^{\times}$ > 0

where s_N is the mean stress for nucleation.

We shall conclude this survey by considering on of the important concepts in the theory of continuum models in porous ductile solids; The stress-strain Relations.

(c) The stress-strain Relations:

In our discussion, we shall assume that the total strain is the sum of the elastic and plastic part. Gurson (1977a) presented a model in which the plastic part of the macroscopic strain-rate for porous ductile material is given by

$$'_{ij} = \Lambda \frac{\partial \Phi}{\partial \sigma^{ij}} \tag{27}$$

Since normality of the plastic flow rule the matrix material implies macroscopic normality, we assume that equation (26) also applies during nucleation of new voids. The microscopic equivalent tensile flow stress s_m and equivalent plastic strain are reported to be averages that do not specify the actual microscopic fields in the matrix material surrounding the voids. We assume that the rate of equivalent plastic work in the matrix material equals the macroscopic rate of plastic work.

$$\sigma^{ij} \iota_{ij}^p = (1-f)\sigma_m \dot{\in}_m^p \tag{27}$$

If we now use equation (24) for the value of $\dot{\in}_m^p$, we obtain the following expression for the rate of the equivalent tensile flow stress in the matrix material

$$\sigma_m = \frac{EE_t}{E-E_t} \cdot \frac{\sigma^{ij}\ \iota_{ij}^p}{(1-f)\dot{\sigma}_m} \tag{28}$$

If we apply the consistency condition that $\dot{\Phi}$ = 0 which is needed during plastic loading, the value ^ of can be obtain from equation (26). If we now use equation (21),(22), (23) and (24) with the consistency condition and we apply equation (26), we obtain the following set equations.

$$\dot{\iota}_{ij}^p = \frac{1}{H} m_{ij}^G m_{kl}^G m_{kl}^F \overset{\nabla}{\sigma}{}^{kl} \tag{29}$$

where $\overset{\nabla}{\sigma}{}^{kl}$ had been defined using equation (13) and

$$m_{ij}^G = {}^3\!/_2 \frac{s_{ij}}{\sigma_m} + \alpha G_{ij} m_{ij}^F = {}^3\!/_2 \frac{s_{ij}}{\sigma_m} \beta G_{ij} \tag{30}$$

$$\alpha \frac{f^*}{2} q_1 q_2 \sinh \frac{q_1}{2} \frac{\sigma_k^k}{\sigma_m}, \beta = \alpha + \frac{B\sigma_m}{6} \frac{\partial \Phi}{\partial f} \tag{31}$$

$$H = \frac{\sigma_m}{2}\left[-3\alpha(1-f)\frac{\partial \Phi}{\partial f} - \left(\frac{\partial \Phi}{\partial f} A + \frac{\partial \Phi}{\partial \sigma_m}\right)\frac{EE_t}{E-E_t}\frac{1}{(1-f)}\left(\frac{\sigma_e^2}{\sigma_m^2} + \alpha \frac{\sigma_k^k}{\sigma_m}\right)\right]$$

It must be stated here that the concepts discussed in this paper may be applied also to theory of continuum Damage Mechanics. Continuum damage models for ductile metals or generally based on micromechanical considerations. Feucht, Gross and Wriggers (1997) developed a non local version of Gurson's model for damage of ductile metal, the fundamental principle of which is based on the concepts already discussed in this paper.

Concluding Remark:

The source of motivation for the paper is the celebrated work of Tvergaard (1990). We have surveyed some basic concepts governing contnuum model of porous ductile solids. We are aware that there are other important concepts that needs attention, we have concentrated on the fundamental ones that form the foundation for other concepts. Some of these concepts discussed here are also available in some standard text books on continuum Mechanics or Plasticity.

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