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A Computational Method to the One-dimensional Convection-diffusion Equation Subject to a Boundary Integral Condition with an Error Estimation

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Abstract: In this research a numerical technique is developed for the one-dimensional convection-diffusion equation with classical and integral boundary conditions. A new iterative process has been presented for the numerical solution of this kind of partial differential equation. An efficient error estimation for this method is also introduced. A simple and efficient algorithm for numerical solution of the method introduced.

Key words: Convection-diffusion equation; Nonlocal boundary conditions; Error estimation; Parabolic equation.

INTRODUCTION

In 1963, nonlocal boundary equation have been presented by Cannon, (1963) and Batten, (1963) independently. Then, parabolic initial-boundary problems with nonlocal integral conditions for parabolic equations were investigated by Kamynin, (1964) and Ionkin, (1977). Then, many numerical methods are applied to solve this case of problem.

In this paper a new matrix formulation is presented for the problem of obtaining numerical approximations to $u(x,t)$ which satisfies the convection-diffusion equation:

$$u_t - \alpha u_{xx} + \beta u_x = f(x,t), \quad x\in(0,l), \quad 0<t<T, \tag{1}$$

with the initial condition

$$u(x,0) = p(x), \quad 0 \le x \le l, \tag{2}$$

and boundary condition

$$\frac{\partial u}{\partial x}u(1,t) = g(t), \quad 0 < t \le T, \tag{3}$$

And

$$\int_0^1 \mathrm{k(x)u(x,t)dx} = m(t), \quad 0 < t \le T \tag{4}$$

where the functions $f(x,t)$, $p(x)$, $k(x)$, $g(t)$ and $m(t)$ and the constants b, α and β are known. Problems of these types arise in the quasi-static theory of thermolasticity.

Recently, expansion methods based on various polynomials have been utilized for different equations (Saadatmandi and Dehghan, Tari and Shahmorad, 2008; Hosseini and Shahmorad, 2005). The theoretical discussion of these case of equations can be found in (Kamynin 1964; Tari and Shahmorad, 2008). The famous work of (Ding and Zhangb, 2009) was one of the first to the solution of similar parabolic inverse problems.

Authors of (Dehghan, 2007; Tatari and Dehghan, 2009; Dehghan, 2005; Smith, 1998) considered this case of equations numerically. In (Ding and Zhangb, 2009), a semi-discrete and a pade approximation method to

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propose a new difference scheme for solving equation (1) with classical conditions have been presented. There has been much work on computing a finite difference approximation solution of equation (1) (Smith, 1998; Strikwerda, 1989). Similar problems can be found at (Soltanalizadeh, DOI:10.1016/j.optcom.2010.12.074.) and (Roohani Ghehsareh, Soltanalizadeh, Abbasbandy,DOI:10.1080/00207160.2010.521816.)

Description of method:

In (1 – 4), the functions $f(x,t)$, $p(x)$, $k(x)$, $g(t)$ and $m(t)$ generally are not polynomials. We assume that these functions are polynomial or they can be approximated by polynomials to any degree of accuracy. So we can write:

$$\begin{cases} f(x,t) \approx \sum_{i=0}^{n}\sum_{j=0}^{n} f_{ij}x^i t^j = X^T F T, \\ k(x) \approx \sum_{i=0}^{n} k_i x^i = X^T K, \\ p(x) \approx \sum_{i=0}^{n} p_i x^i = X^T P, \\ m(t) \approx \sum_{j=0}^{n} m_j t^j = MT, \\ g(t) \approx \sum_{j=0}^{n} g_j t^j = GT, \end{cases} \tag{5}$$

Where

$$\begin{cases} X = [1, x, x^2, \ldots, x^n]^T, \\ T = [1, t, t^2, \ldots, t^n]^T, \\ K = [k_0, k_1, k_2, \ldots, k_n]^T, \\ P = [p_0, p_1, p_2, \ldots, p_n]^T, \\ G = [g_0, g_1, g_2, \ldots, g]^T, \\ M = [m_0, m_1, m_2, \ldots, m_n]^T, \\ F = [f_0, f_1, f_2, \ldots, f_n]^T, \\ F_i = [f_{0i}, f_{1i}, f_{2i}, \ldots, f_{ni}]^T, \qquad i = 0,1,\ldots,n \end{cases} \tag{6}$$

Therefore we consider approximate solution of the form

$$U_n(x,t) \approx \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij}x^i t^j = X^T U T, \tag{7}$$

Where

$$U = [U_0, U_1, U_2, \ldots, U_n], \tag{8}$$

With

$$U_i = [u_{0i}, u_{1i}, u_{2i}, \ldots, u_{ni}]^T.$$

The matrix U is an $(n+1)\times(n+1)$ matrix which contains $(n+1)^2$ unknown coefficients of $u_n(x,t)$ To find these unknowns, we proceed as follows.

We first consider the initial condition

$$u(x,0) = a(x), \qquad x \in [0,1].$$

then by substituting from (5) and (7), we obtain

$$X^T U_0 = X^T A,$$

which implies

$U_0=A$,

since X is a basis vector.

From (9), we can find the first column of U.

Now we consider the nonlocal boundary condition

$$\int_0^1 k(x)u(x,t)dx = g(t), \quad 0 < t \le T$$

We substitute again from (5) and (7) and obtain

$$\sum_{j=0}^{n} g_j t^j = \int_0^1 \left(\sum_{h=0}^{n} k_h x^h\right)\left(\sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij} x^i t^j\right) dx$$

$$= \left(\sum_{h=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{n} k_h u_{ij} t^j \int_0^1 x^{h+i}\, dx\right)$$

$$= \left(\sum_{h=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{n} k_h u_{ij} t^j v_{h+i+1}\right), \quad 0 < t \le T.$$

or equivalently

$$DUT=GT \tag{12}$$

Where $D = [d_0, d_1, d_2, \dots, d_n]^T$, with

$$d_i = \sum_{h=0}^{n} k_h v_{h+i+1}, \qquad i = 0,1,2,\dots,n,$$

And

$$v_{h+i+1} = \frac{1}{1+h+i}, \qquad i = 0,1,2,\dots,n,$$

The equivalent from (12) is

$$DU=G. \tag{13}$$

Since T is a basis vector.

We use the following lemma to write equation (1) in the matrix form to determine remainder equations.

Lemma 2.1:

Let $y_n(x) = X^T Y T$, with $X = [1, x, x^2, \dots, x^n]^T$, $T = [1, t, t^2, \dots, t^n]^T$ and $Y = [y_0, y_1, y_2, \dots, y_n]^T$, $Y_j = [y_{0j}, y_{1j}, y_{2j}, \dots, y_{nj}]$,

Then

$$\begin{cases} \frac{d^r}{dx^r} y_n(x,t) = X^T (\eta^T)^r YT, \\ \frac{d^r}{dt^r} y_n(x,t) = X^T Y \eta^r T. \end{cases} \tag{14}$$

Where

$$\eta = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}_{(n+1)(n+1)}$$

Corollary 2.2:

By using lemma 2.1, we have

$$\begin{cases} u_{xx} = X^T (\eta^T)^2 YT \\ u_x = X^T \eta YT \\ u_t = X^T Y \eta T \end{cases} \tag{15}$$

Therefore substituting from (5), (7) and (15) in (1), leads to

$$X^T U \eta T - \alpha X^T U \eta^2 T + \beta X^T \eta YT = X^T FT,$$

then

$$U\eta - \alpha U \eta^2 + \beta \eta U = F, \tag{16}$$

Applying the method:

In this section, we arrange the linear equations obtained in the previous section, to have a system of $(n+1)(n+1)$equations for the $(n+1)(n+1)$ unknowns as follows:
By equation (9), we have

$$u_{i0} = p_i, \ \ i = 0,1,2 \ldots, n, \tag{17}$$

and from (12), we get

$$\sum_{i=1}^{n} i u_{ij} = g_j, \ \ j = 1,2,\ldots,n, \tag{18}$$

from (15), we have

$$\sum_{i=0}^{n} d_i u_{ij} = m_j, \ \ j = 1,2,\ldots,n, \tag{19}$$

Finally, we select remainder equations from (16). then for j=1,2...,n, we have

$$\begin{cases} u_{ij} = \frac{1}{j}\left(f_{i,j-1} + \beta(i+1)u_{i+1,j-1} - \alpha(i+2)(i+1)u_{i+2,j-1}\right), \\ \qquad\qquad i = 2,\ldots,n-2, \\ \qquad u_{n-1,j} = \frac{1}{j}\left(f_{n-1,j-1} + \beta(i+1)u_{n,j-1}\right). \end{cases} \tag{20}$$

Then now by the equations (17-21), we can the numerical computations for various problems.

Error estimation:

We know that an orthogonal bases has the advantage that it guarantees convergence of the method based on it. Finding error estimation of a numerical method is an important step of it. S. Shahmorad et al. found various error estimation for expansion methods. Similar their method, we estimate the size of error.

For this end, if we let $u(x,t)$ is the exact solution and $u_{n,m}(x,t)$ be the approximate, then we define

$$e(x,t) = u(x,t) - u_{n,m}(x,t)$$

as the error function.

Then by substituting $u_{n,m}(x,t)$ in (1), we get

$$u_t^{n,m}(x,t) - \alpha u_{xx}^{n,m}(x,t) + \beta u_x^{n,m}(x,t) = f(x,t) + h^{n,m}(x,t) \tag{21}$$

With

$$\begin{cases} u^{n,m}(x,0) = p(x), \quad 0 \le x \le l \\ u_x^{n,m}(1,t) = g(t), \quad 0 < t \le T \\ \int_0^1 K(x) u^{n,m}(x,t)dx = m(t), \; 0 < t \le T \end{cases} \tag{22}$$

Then $h^{n,m}(x,t)$ will be the perturbation term that can be found to under Formula

$$h^{n,m}(x,t) = u_t^{n,m}(x,t) - \alpha u_{xx}^{n,m}(x,t) + \beta u_x^{n,m}(x,t) - f(x,t) \tag{23}$$

By subtracting (23) from (21), we have

$$e_t^{n,m}(x,t) - \alpha e_{xx}^{n,m}(x,t) + \beta e_x^{n,m} = -h^{n,m}(x,t) \tag{24}$$

with the new conditions

$$\begin{cases} e^{n,m}(x,0) = 0, \quad 0 \le x \le l \\ e_x^{n,m}(1,t) = 0, \quad 0 < t \le T \\ \int_0^1 K(x) u^{n,m}(x,t)dx = m(t), \; 0 < t \le T \end{cases} \tag{25}$$

Then (23) and (25) give us an error estimation for error function .

Numerical Examples:

In this section, we solve under examples by using the presented method .

Example 1:

Consider the heat equation

$$u_t - u_{xx} + u_x = -3 + 3t - 2t^2 + 5x - 2tx^2 - 2xt^2, \qquad x \in (0,1), \qquad 0 < t < T,$$

with the conditions

$$\begin{cases} u(x,0) = x, \quad 0 < t \le T, \quad 0 \le x \le 1, \\ \frac{\partial u}{\partial x} u(1,t) = 5 + t - 2t^2, \quad 0 < t \le T, \\ \int_0^1 xu(x,t) = \frac{5}{6} + \frac{1}{3}t + \frac{1}{4}t^2, \quad 0 < t \le T, \end{cases}$$

By using this method and choosing n = 2, we obtain

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

and by using (7), we get $u(x,t) = t^2 + x + xt + 2x^2 + t^2x^2$ which is the exact solution.

Example 2:

Consider the heat equation

$$u_t - u_{xx} = (x^2 - 2)e^t, \ x \in (0,1), \quad 0 < t < T,$$

with the conditions

$$\begin{cases} u(x,0) = x^2, \quad 0 < t \le T, \quad 0 \le x \le 1, \\ \frac{\partial u}{\partial x} u(1,t) = 2xe^t, \quad 0 < t \le T, \\ \int_0^1 u(x,t) = \frac{e^t}{3}, \quad 0 < t \le T, \end{cases}$$

This example has analytical solution $u(x,t) = x^2 e^t$. The computed results at various time lengths with are shown in Table 1.

Table 1: Absolute errors of the presented method for

t	n=5	n=10	n=15
0.0	0	0	0
0.1	3.522E-10	1.110E-16	1.110E-16
0.2	2.287E-8	1.665E-16	5.551E-17
0.3	2.644E-7	1.144E-14	5.551E-17
0.4	1.508E-6	2.718E-13	5.551E-17
0.5	5.839E-6	3.191E-12	1.110E-16
0.6	1.770E-5	2.391E-11	2.776E-16
0.7	4.532E-5	1.315E-10	2.554E-15
0.8	1.026E-4	5.762E-10	1.732E-14
0.9	2.112E-4	2.124E-9	9.825E-14
1.0	4.038E-4	6.828E-9	4.634E-13

Conclusions:

In this article we presented a computational method for solving the parabolic convection-diffusion equation with an integral condition. By using a simple algorithm that it have high accuracy. (see the example 2). If the exact solution of (1-4) be a polynomial of degree, then by matrix formulation method, we find the exact solution for (see example 1). By the other expansion methods, authors obtain a system of algebraic equations and then solve it, but in this paper we suggest an interesting process with a simple algorithm.

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