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# A Slacks-base Measure of Super-efficiency for Dea with Negative Data

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**Abstract:** Data envelopment analysis (DEA) is a non-parametric method for measuring the efficiency of a set of decision making units (DMUs), such as companies or public sector agencies. The main DEA models are only used for positive data. In recent years, some models have been presented to deal with negative data in DEA models. However, these models do not discriminate between efficient DMUs and only evaluate them as being efficient. In this paper, we propose a model by which we discriminate between such DMUs. Then, using the slacks-based measures (SBM) of efficiency introduced by Tone in 2001, we extend the super-efficiency problem for DEA models with positive and negative inputs and outputs. We, then, discuss the model and its stability, and elaborate on the problem by a numerical example.

**Key words:** Data envelopment analysis, Super efficiency, Slack-based model, Negative data in DEA.

## INTRODUCTION

In DEA models (charnes *et al*. (1978), Cooper *et al*. (2000)), all the DMUs that have the best performance are assigned the efficiency score of unity. This is also true about DEA models with negative data (Scheel(2001), portelal *et al*. (2004), Emrouznejad *et al*. (2010)). There are therefore many DMUs with efficiency scores of unity. Some models have been presented to discriminate between and rank these units when the inputs and outputs are positive. For example, see Anderson and Peterson(1993), Doyle and green (1993, 1994), Stewart (1994), Tofallis (1996), Seiford and Zhu (1999), and Zhu (2001), Tone (2002). This problem is called the super-efficiency problem. In conventional DEA models, the data are assumed to be positive. In many applications, such as loss when net profit is an output variable, negative inputs and outputs emerge. In recent years, some papers have dealt with negative data. Scheel (2001) provided an estimate cope with negative data in DEA models, in which the absolute value of negative outputs and inputs are considered as inputs and outputs, respectively. The additive model (Charnes *et al*. (1998)) under the variable returns to scale (VRS) assumption can be used to deal with negative data, since this model is translation in variant and a fixed value can be added to all inputs and outputs so that they are replaced by positive values. This is while the additive model does not specify the amount of efficiency and the results obtained are dependent on the unit of measurement of the inputs and outputs. Sharp *et al*. (2006) presented a modified slacks-based measure (MSBM), which had some limitations, such as the requirement to have at least one positive input and output. Portela *et al*. (2004) provided a model that specified a measure of efficiency for each DMU. The inputs and outputs could be negative. This model did not guarantee the existence of projections on the Pareto efficiency frontier. Emrouznejad *et al*. (2010) have proposed a semi-oriented radial measure (SORM) to handle negative data and overcome the shortcomings of previous models. This model can be employed for models with both negative input and output. While this model increases the dimension of the problem and eliminates a part of the initial PPS, the targets that are obtained by the model are not worse than the observed surfaces. In all of the existing models, the DMUs that are evaluated as efficient have the efficiency score of unity and no mention is made of better performance or ranking of the units. In the present paper, we use the slacks-based measure of efficiency (Tone, 2001) and the measure of super-efficiency (Tone, 2002) to propose a model that is able to rank all extreme efficient DMUs with positive and negative inputs and outputs and obtain the amount of super-efficiency. The paper is organized as follows. In Section 2, we discuss the super-efficiency model

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provided by Tone, 2002, and then present the super-efficiency model for positive and negative data. The model is extended for the input and output orientations and its feasibility and stability conditions are discussed in Section 3. We extend the model for the case in which inputs and outputs assume zero values, in Section 4, and then elaborate on the problem by numerical examples.

***Super-efficiency Evaluated by SBM:***

Consider a set of n observed DMUs, $\{DMU_j | j = 1, \dots, n\}$ , The input and output matrices corresponding to these DMUs are $X = (x_{ij}) \in R^{m \times n}$ and $Y = (y_{rj}) \in R^{s \times n}$ respectively. Suppose X>0, Y>0. Tone (2001) introduced the super-efficiency model corresponding to $DMU_o$ as follows.

$$\begin{aligned} Min \quad & \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r / y_{ro}} \\ s.t \quad & \sum_{j=1, j \neq o}^{n} \lambda_j x_{ij} \leq \bar{x}_i, \quad i = 1, \dots, m, \\ & \sum_{j=1, j \neq o}^{n} \lambda_j y_{rj} \geq \bar{y}_r, \quad r = 1, \dots, s, \\ & \sum_{j=1, j \neq o}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \dots, n, \\ & x_{io} \leq \bar{x}_i, \quad i = 1, \dots, m, \quad \bar{y}_r \leq y_{ro}, \quad r = 1, \dots, s. \end{aligned} \tag{1}$$

The above model is used for ranking extreme efficient units. The greater the objective function value, the higher the rank of the corresponding extreme efficient DMUs. The model indicates the distance $DMU_o$ of the points $(\bar{x}, \bar{y})$ in the new production possibility set (PPS) after the removal of $DMU_o$ regarding $\bar{x} \geq x_o$ and $\bar{y} \leq y_o$ and considers the point that has the smallest distance from the DMU under evaluation. As can be seen, the objective function value of Problem (1) is greater than or equal to 1, since the numerator is greater than 1 and the denominator is less than 1. It should be noted that the VRS assumption of technology must be used with negative data since the constant returns to scale (CRS) assumption those not hold for negative data. For instance, consider DMUs A and B. If the activity vectors are of the form $(x_1, y_1, y_2)$ and the corresponding vectors of these DMUs are A=(1,1,1) and B=(1,-2,3), both DMUs are CCR-efficient because the first output of A is greater than that of B and the second output of B is greater than that of A. If we assume constant returns to scale, the vector 0.5B=(0.5,-1,1.5) belongs to the PPS and dominates vector A, which cannot be true. It should be noted that in the numerator of the objective function of Model (1), concerning the constraints, we have $\frac{\bar{x}_i}{x_{io}} \geq 1$, $i = 1, \dots, m$ . Also, in the denominator of the objective function, we have $\frac{\bar{y}_r}{y_{ro}} \leq 1$ , $r = 1, \dots, s$. If the input and output matrix includes negative elements, the above ratios cannot be used in Model (1). For example, consider four DMUs with the activity vector $(x_1, y_1)$ , where $x_1$ can assume negative values. These DMUs are displayed in Figure 1. b. The PPS boundary includes the line segment segments, AB, BC, CD, the line that starts from D and is parallel to the input axis and the line that starts from A and is parallel to the output axis. The new PPS after excluding $DMU_B$ includes the line segments, AC, CD, the line that starts from D and is parallel to the input axis and the line that starts from A and is parallel to the output axis. To determine the set of points $(\bar{x}_1, \bar{y}_1)$ in the new PPS, regarding $(\bar{x}_1, -\bar{y}_1) \geq (x_{1B}, -y_{1B})$, that have the shortest distance from $DMU_B$ any of the points of the line segment EF can be considered. If we use the ratio $\frac{\bar{x}_1}{x_{1B}}$ in the objective function, it will have a value less than one, since $|x_{1B}| \geq |\bar{x}_1|$ . In this case the ratio $\frac{x_{1B}}{\bar{x}_1} \geq 1$ can be used as the desirable ratio in the numerator of the objective function. As a result, if the set of points $(\bar{x},\bar{y})$ in the new PPS after the elimination of the DMU under assessment, with the constraint $(\bar{x},-\bar{y}) \geq (x_o,-y_o)$ , consists of the points $x_{io} \leq \bar{x}_i < 0$ (for some input components), then we can employ the ratio $\frac{x_{io}}{\bar{x}_i} \geq 1$ in the numerator of the objective function as the desirable ratio.

However, if the set of points $(\bar{x},\bar{y})$ in the new PPS, with the constraint $(\bar{x},-\bar{y}) \geq (x_o,-y_o)$ only includes the points $0 < x_o \leq \bar{x}$ (for all input components), then we can use the ratio $\frac{\bar{x}_i}{x_{io}} \geq 1$ in the numerator of the objective function.

Now, consider the case where the activity vectors of the DMUs are of the form $(x_1, y_1, y_2)$ . The four observed DMUs A, B, C, and D are displayed in Figure 1. a. The efficiency frontier includes the segments, AB, BC, CD, the line that starts from D and is parallel to the second output axis and the line that starts from A and is parallel to the first output axis. In this case, the set of points $(\bar{x}_1, \bar{y}_1, \bar{y}_2)$ in the new PPS after eliminating $DMU_B$ , regarding $(\bar{x},-\bar{y}_1,-\bar{y}_2) \geq (x_B,-y_{1B},-y_{2B})$ includes the points of the line segment EF. Here, since $|y_{1B}| \leq |\bar{y}_1|$ the ratio $\frac{\bar{y}_1}{y_{rB}} \geq 1$ cannot be used in the denominator of the objective function of Model (1), since it has a value greater than one. Thus, the ratio $0 < \frac{y_{1B}}{\bar{y}_1} \leq 1$ can be used in the denominator of the objective function as the desirable ratio to produce a ratio less than one. Therefore, if the set of points $(\bar{x},\bar{y})$ in the new PPS after the exclusion of the DMU under assessment, regarding $(\bar{x},-\bar{y}) \geq (x_o,-y_o)$ , consists of the points $\bar{y}_r \leq y_{ro} < 0$ , (for some output components) then we can use the ratio $0 < \frac{y_{ro}}{\bar{y}_r} \leq 1$ as the desirable ratio in the denominator of the objective function of Model (1). However, in the case where the set of points $(\bar{x},\bar{y})$ in the new PPS, considering $(\bar{x},-\bar{y}) \geq (x_o,-y_o)$ only includes the pointsthe ratio $0 < \frac{\bar{y}_r}{y_{ro}} \leq 1$ can be employed as the desirable ratio in the denominator of the objective function.

Now, assume that the activity vectors of four DMUs A, B, C, D are of the form $(x_1, y_1)$ , as shown in Figure 1. b. The set of points $(\bar{x}_1, \bar{y}_1)$ , in the new PPS after eliminating $DMU_B$ , regarding $(\bar{x}_1,-\bar{y}_1) \geq (x_{1B},-y_{1B})$ includes the line segment DE. In this case, we can have $x_{1B} < 0$ and $\bar{x}_1 > 0$ , which cannot be used in the numerator of the objective function to produce a ratio greater than one. However, there exist points of the line segment DE for which $x_B$ and $\bar{x}_1$ are of the same sign and $x_B \leq \bar{x}_1 < 0$. In this case, we can add the constraint $x_B \leq \bar{x}_1 < 0$ to the model for evaluating $DMU_B$ , so that the ratio $1 \leq \frac{x_{1B}}{\bar{x}_1}$ can be employed in the objective function. As a result, if the set of points $(\bar{x},\bar{y})$ in the new PPS after excluding the DMU under evaluation, regarding $(\bar{x},-\bar{y}) \geq (x_o,-y_o)$ consists of the points such that for some input components we have $\bar{x}_i > 0$ and $x_{io} < 0$, then we can add the constraint $x_{io} \leq \bar{x}_i < 0$ to the model for

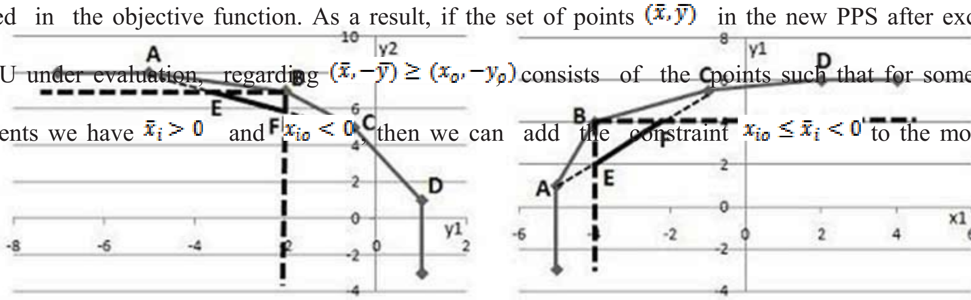

**Fig. 1:** a) The pps in the space outputs b) The pps in the space input and output.

evaluating $DMU_o$ , as there exists points of the new PPS for which $x_{io} \leq \bar{x}_i < 0$ . By adding this constraint for the negative components, the ratio $1 \leq \frac{x_{io}}{\bar{x}_i}$ can be used in the numerator of the objective function as the desirable ratio.

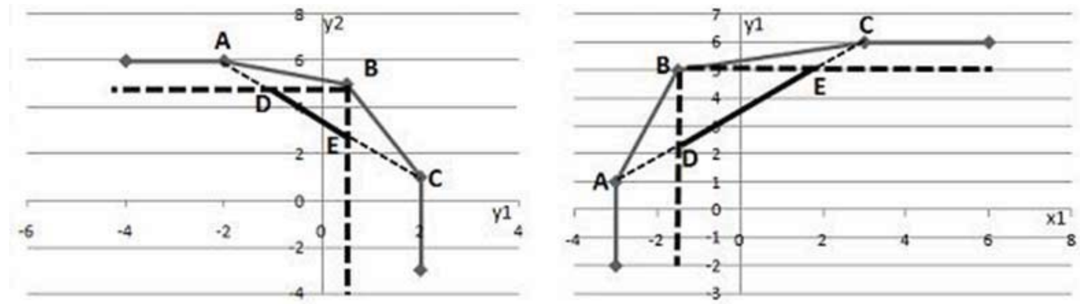


**Fig. 2:** a)The pps in the space outputs b)The pps in the space input and output.

Now, consider the case in which the activity vectors of the DMUs are of the form $(x_1, y_1, y_2)$ . Four DMUs A, B, C, and D, which consume the same inputs, are shown in Figure 2. a. The set of points $(\bar{x}_1, \bar{y}_1, \bar{y}_2)$ in the new PPS after eliminating $DMU_B$ , considering $(\bar{x}_1, -\bar{y}_1, -\bar{y}_2) \geq (x_{1B}, -y_{1B}, -y_{2B})$ includes the line segment DE. In this case, we can have $\bar{y}_1 < 0$ and $0 < y_{1B}$ which cannot be used in the denominator of the objective function to produce a positive ratio less than one. But there exist points of the line segment DE for which $\bar{y}_1$ and $y_{1B}$ are of the same sign and $0 < \bar{y}_1 \leq y_{1B}$ . So we add the constraint $0 < \bar{y}_1 \leq y_{1B}$ to the model to use the ratio $0 < \frac{y_{1B}}{\bar{y}_1} \leq 1$ in the denominator of the objective function as the desirable ratio. Therefore, if the set of points $(\bar{x}, \bar{y})$ in the new PPS after eliminating the DMU under assessment includes the points $(\bar{x}, -\bar{y}) \geq (x_o, -y_o)$ and for some output components we have $\bar{y}_r < 0$ and $0 < y_{ro}$ , then we can add the constraint $0 < \bar{y}_r \leq y_{ro}$ to the model for evaluating $DMU_o$ , since there exist points of the new PPS such that $0 < \bar{y}_r \leq y_{ro}$ . By adding the above constraint, we can employ the ratio $0 < \frac{\bar{y}_{ro}}{\bar{y}_r} \leq 1$ in the denominator of the objective function as the desirable ratio.

Suppose $x_{ij} \neq 0$ , $y_{rj} \neq 0$ , $i = 1, \dots, m,\ r = 1, \dots, s,\ j = 1, \dots n$ , and define $I = \{i | x_{io} < 0\}$ , $O = \{r | y_{ro} < 0\}$ , $\bar{I} = \{i | x_{io} > 0\}$ , $\bar{O} = \{r | y_{ro} > 0\}$ . With regard to the above discussion, the model for obtaining the super-efficiency of the DMU under evaluation, $DMU_o$ , is proposed as follows.

$$
\begin{aligned}
Min \quad & \frac{\frac{1}{m}\left(\sum_{i\in I}\frac{\bar{x}_i}{x_{io}}+\sum_{i\in \bar{I}}\frac{x_{io}}{\bar{x}_i}\right)}{\frac{1}{s}\left(\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}}+\sum_{r\in \bar{O}}\frac{y_{ro}}{\bar{y}_r}\right)} \\
s.t \quad & \sum_{j=1,j\neq o}^{n} \lambda_j x_{ij} \leq \bar{x}_i, \qquad i=1,\dots,m, \\
& \sum_{j=1,j\neq o}^{n} \lambda_j y_{rj} \geq \bar{y}_r, \qquad r=1,\dots,s, \\
& \sum_{j=1,j\neq o}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \qquad j=1,\dots,n, \\
& 0 < x_{io} \leq \bar{x}_i \quad i\in I, \quad x_{io} \leq \bar{x}_i < 0 \quad i \in \bar{I}, \\
& 0 < \bar{y}_r \leq y_{ro} \quad r\in O, \quad \bar{y}_r \leq y_{ro} < 0 \quad r \in \bar{O}.
\end{aligned}
\tag{2}
$$

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The above model is a nonlinear model and cannot be linearized by using the necessary transformations. In this respect, we present the following model, in which the ratio of the negative outputs is used in the numerator and the ratio of the negative inputs is used in the denominator of the objective function.

Suppose $|I| = m_1$ and $|O| = s_1$ denote the number of elements of sets *i* and *o*, respectively. To present the new model, we assume $m_1 + s - s_1 \neq 0$, that is we have at least one positive input or one negative output. Similarly, we consider $m - m_1 + s_1 \neq 0$ , that is there are at least one positive output or one negative input corresponding to the input and output components of $DMU_o$ . The new model is provided as follows.

$$
\begin{aligned}
[SuperSBMN] \quad \delta^* = Min \quad & \frac{\frac{1}{m_1+s-s_1}\left(\sum_{i\in I}\frac{\bar{x}_i}{x_{io}}+\sum_{r\in \bar{O}}\frac{\bar{y}_r}{y_{ro}}\right)}{\frac{1}{m-m_1+s_1}\left(\sum_{i\in \bar{I}}\frac{\bar{x}_i}{x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}}\right)} \\
s.t \quad & \sum_{j=1,j\neq o}^{n} \lambda_j x_{ij} \leq \bar{x}_i, \qquad i=1,\dots,m, \\
& \sum_{j=1,j\neq o}^{n} \lambda_j y_{rj} \geq \bar{y}_r, \qquad r=1,\dots,s, \\
& \sum_{j=1,j\neq o}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \qquad j=1,\dots,n, \\
& 0 < x_{io} \leq \bar{x}_i \quad i\in I, \quad x_{io} \leq \bar{x}_i < 0 \quad i \in \bar{I}, \\
& 0 < \bar{y}_r \leq y_{ro} \quad r\in O, \quad \bar{y}_r \leq y_{ro} < 0 \quad r \in \bar{O}.
\end{aligned}
\tag{3}
$$

The above model is a nonlinear model. To transform it to a linear one, we can use the Charnes-Cooper transformation. We set:

$\frac{1}{m-m_1+s_1}\left(\sum_{i\in \bar{I}}\frac{\bar{x}_i}{x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}}\right)=\frac{1}{t}$ . Then, we will have: $\frac{1}{m-m_1+s_1}\left(\sum_{i\in \bar{I}}\frac{t\bar{x}_i}{x_{io}}+\sum_{r\in O}\frac{t\bar{y}_r}{y_{ro}}\right)=1$ .

We put $\tilde{x}_i = t\bar{x}_i \quad i=1,\dots,m$ , $\tilde{y}_r = t\bar{y}_r \quad r=1,\dots,s$ , and $\mu_j = t\lambda_j \quad j=1,\dots,n$ .

By the above changes of variables, Model (3) is transformed into the following model:

$$
\begin{aligned}
\tau^* = Min \quad & \frac{1}{m_1+s-s_1}\left(\sum_{i\in I}\frac{\tilde{x}_i}{x_{io}}+\sum_{r\in \bar{O}}\frac{\tilde{y}_r}{y_{ro}}\right) \\
s.t \quad & \frac{1}{m-m_1+s_1}\left(\sum_{i\in \bar{I}}\frac{\tilde{x}_i}{x_{io}}+\sum_{r\in O}\frac{\tilde{y}_r}{y_{ro}}\right)=1 \\
& \sum_{j=1,j\neq o}^{n} \mu_j x_{ij} \leq \tilde{x}_i, \qquad i=1,\dots,m, \\
& \sum_{j=1,j\neq o}^{n} \mu_j y_{rj} \geq \tilde{y}_r, \qquad r=1,\dots,s, \\
& \sum_{j=1,j\neq o}^{n} \mu_j = t, \quad \mu_j \geq 0, \qquad j=1,\dots,n, \qquad t>0, \\
& 0 < t x_{io} \leq \tilde{x}_i \quad i\in I, \quad t x_{io} \leq \tilde{x}_i < 0 \quad i \in \bar{I}, \\
& 0 < \tilde{y}_r \leq t y_{ro} \quad r\in O, \quad \tilde{y}_r \leq t y_{ro} < 0 \quad r \in \bar{O}.
\end{aligned}
\tag{4}
$$

Suppose the solution of the above LP is $(\tau^*, \mu^*, \tilde{x}^*, \tilde{y}^*, t^*)$ , then the optimal solution corresponding to the super SBMN problem is obtained as follows.

$\delta^* = \tau^*$ , $\lambda_j^* = \frac{\mu_j^*}{t^*},\ j=1,\dots,n$ , $\bar{x}_i^* = \frac{\tilde{x}_i^*}{t^*}$ , $i=1,\dots,m,$ $\bar{y}_r^* = \frac{\tilde{y}_r^*}{t^*}$ , $r=1,\dots,s$ .

***Theorem 1:*** Suppose $\alpha \leq 1, \beta \geq 1$ and $\hat{x}_i = \alpha x_{io} i \in$, $\hat{x}_i = \beta x_{io} i \in$, $\hat{y}_r = \beta y_{ro} r \in O$, $\hat{y}_r = \alpha y_{ro} r \in \bar{O}$ re the components of a DMU ($(\hat{x}, \hat{y})$) with inputs and outputs respectively decreased and increased with regard to $DMU_o$. Then, the super-efficiency value corresponding to $(\hat{x}, \hat{y})$ will not be less than the super-efficiency value corresponding to $(x_o, y_o)$ .

**Proof.** The super-efficiency value ($\hat{\delta}^*$ ) corresponding to $(\hat{x}, \hat{y})$ is obtained by solving the following model.

$[SuperSBMN]'$

$$\hat{\delta}^* = Min \quad \frac{\frac{1}{m_1+s-s_1}(\sum_{i\in I}\frac{\bar{x}_i}{\alpha x_{io}}+\sum_{r\in \bar{O}}\frac{\bar{y}_r}{\alpha y_{ro}})}{\frac{1}{m-m_1+s_1}(\sum_{i\in \bar{I}}\frac{\bar{x}_i}{\beta x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{\beta y_{ro}})} = Min \quad \frac{\beta}{\alpha} \quad \frac{\frac{1}{m_1+s-s_1}(\sum_{i\in I}\frac{\bar{x}_i}{x_{io}}+\sum_{r\in \bar{O}}\frac{\bar{y}_r}{y_{ro}})}{\frac{1}{m-m_1+s_1}(\sum_{i\in \bar{I}}\frac{\bar{x}_i}{x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}})}$$

$$\begin{aligned} s.t \quad & \sum_{j=1,j\neq o}^{n} \lambda_j x_{ij} \le \bar{x}_i, \quad i=1,\dots,m, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j y_{rj} \ge \bar{y}_r, \quad r=1,\dots,s, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j = 1, \quad \lambda_j \ge 0, \quad j=1,\dots,n, \\ & 0 < \alpha x_{io} \le \bar{x}_i \quad i \in I, \quad \beta x_{io} \le \bar{x}_i < 0 \quad i \in \bar{I}. \\ & 0 < \bar{y}_r \le \beta y_{ro} \quad r \in O, \quad \bar{y}_r \le \alpha y_{ro} < 0 \quad r \in \bar{O}. \end{aligned} \tag{5}$$

As can be observed, for any feasible solution $(\bar{x},\bar{y})$ of the above model $(\frac{\bar{x}_i}{\alpha} \;\; i\in I,\; \frac{\bar{x}_i}{\beta} \;\; i\in \bar{I},\; \frac{\bar{y}_r}{\beta} \;\; r\in O,\; \frac{\bar{y}_r}{\alpha} \;\; r\in \bar{O}) \in R^{m+s}$ will be a feasible solution for the super SBMN problem. It is true since

$0 < \alpha x_{io} \le \bar{x}_i \quad i\in I \Rightarrow 0 < x_{io} \le \frac{\bar{x}_i}{\alpha} \quad \beta x_{io} \le \bar{x}_i < 0 \quad i \in \bar{I} \Rightarrow x_{io} \le \frac{\bar{x}_i}{\beta} < 0 \quad i \in \bar{I}$.

$0 < \bar{y}_r \le \beta y_{ro} \quad r\in O \Rightarrow 0 < \frac{\bar{y}_r}{\beta} \le y_{ro} \quad r \in O$ . $\bar{y}_r \le \alpha y_{ro} < 0 \quad r \in \bar{O} \Rightarrow \frac{\bar{y}_r}{\alpha} \le y_{ro} < 0 \quad r \in \bar{O}$ .

Then

$\sum_{j=1,j\neq o}^{n} \lambda_j x_{ij} \le \bar{x}_i \le \frac{\bar{x}_i}{\alpha} \quad i \in I$ , $\sum_{j=1,j\neq o}^{n} \lambda_j x_{ij} \le \bar{x}_i \le \frac{\bar{x}_i}{\beta} \quad i \in \bar{I}$ ,

$\sum_{j=1,j\neq o}^{n} \lambda_j y_{rj} \ge \bar{y}_r \ge \frac{\bar{y}_r}{\alpha} \quad r \in O$ , $\sum_{j=1,j\neq o}^{n} \lambda_j y_{rj} \ge \bar{y}_r \ge \frac{\bar{y}_r}{\beta} \quad r \in \bar{O}$ ,

Hence it holds

$$\delta^* \le \frac{\frac{1}{m_1+s-s_1}(\sum_{i\in I}\frac{x_i}{\alpha x_{io}}+\sum_{r\in \bar{O}}\frac{y_r}{\alpha y_{ro}})}{\frac{1}{m-m_1+s_1}(\sum_{i\in \bar{I}}\frac{x_i}{\beta x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{\beta y_{ro}})} = \frac{\beta}{\alpha}\frac{\frac{1}{m_1+s-s_1}(\sum_{i\in I}\frac{x_i}{x_{io}}+\sum_{r\in \bar{O}}\frac{y_r}{y_{ro}})}{\frac{1}{m-m_1+s_1}(\sum_{i\in \bar{I}}\frac{x_i}{x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}})} \tag{6}$$

Since $\frac{\beta}{\alpha} \ge 1$ , By comparing (5) and (6), we have $\delta^* \le \hat{\delta}^*$ . Thus, the efficiency value of $(\hat{x},\hat{y})$ is not less than $(x_o, y_o)$ .

***Theorem 2:***

The super-efficiency value $\delta^*$ (Super SBMN) is unit invariant, i.e., it is independent of the unit of measurement. It should be noted that these units must be the same for all DMUs.

**Proof.** The constraints and the objective function are unit invariant, so the proof is obvious.

Model (3) might be infeasible. For instance, consider two DMUs, by activity vectors $(x_1, y_1)$ . A=(-2,2), B=(3,4). The PPS frontier is specified in Fig. 3. b. After excluding $DMU_A$ , the new PPS is as shown in Figure 3. b. It can be observed that no point $(\bar{x}_1, \bar{y}_1)$ exists in the new PPS such that $x_A \le \bar{x}_1 < 0$ , since for all points in the new PPS we have $\bar{x}_1 > 0$ , thus the corresponding model will be infeasible. In Model (3), this occurs when at the optimal solution of the model we have $\sum_{j=1,j\neq o}^{n} \lambda_j^* x_{ij} > 0$ and $x_{io} < 0$ for some input indices. Since $\sum_{j=1,j\neq o}^{n} \lambda_j^* x_{ij} < \bar{x}_i$ then $\bar{x}_i > 0$ which means that the constraint $x_{io} \le \bar{x}_i < 0$ is not satisfying; thus Model (3) is infeasible.

Now, assume the activity vectors of A and B as B=(1, -2, 4) and A=(1, 1, 1). The efficiency frontier for this case is displayed in Figure 3. a. and the new PPS after eliminating $DMU_A$ is shown in Figure 3. a. As can be seen, there exists no point $(\bar{x}_1, \bar{y}_1, \bar{y}_2)$ in the new PPS such that $0 < \bar{y}_1 \le y_{1A} = 1$ . Since we have $\bar{y}_1 < 0$ for all points in the new PPS; thus, the corresponding model will be infeasible. This occurs in Model (3) in the case when we have $\sum_{j=1,j\neq o}^{n} \lambda_j^* y_{rj} < 0$ and $y_{ro} > 0$ at the optimal solution of Model (3). Since $\sum_{j=1,j\neq o}^{n} \lambda_j^* y_{rj} \ge \bar{y}_r$ then $\bar{y}_r < 0$ , i.e., the constraint $0 < \bar{y}_r \le y_{ro}$ does not hold and, therefore, Model (3) is infeasible.

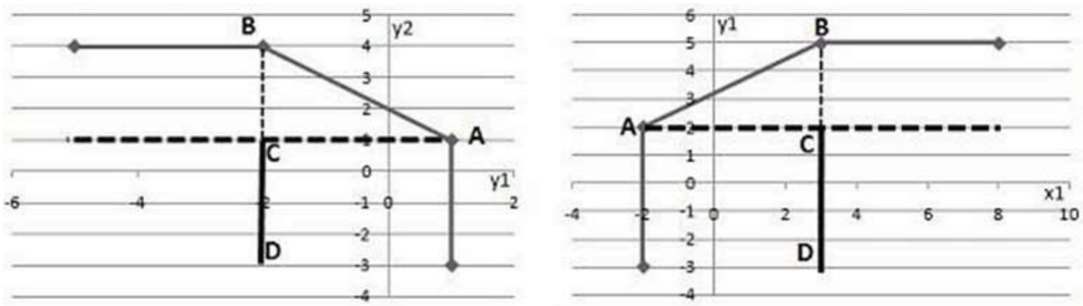


**Fig. 3:** a)The pps in the space outputs b)The pps in the space input and output.

***Theorem 3:***

If there exists a vector $\bar{\lambda} \in R^n$ such that $\sum_{j=1, j\neq o}^{n} \bar{\lambda}_j y_{rj} > 0$ , $\sum_{j=1, j\neq o}^{n} \bar{\lambda}_j x_{ij} < 0$ and $\sum_{j=1, j\neq o}^{n} \bar{\lambda}_j = 1$ for $i = 1, \dots, m,\ r = 1, \dots, s,$ then Model (3) is feasible.

**Proof.** The vector $\bar{\lambda}$ can be selected such that $\sum_{j=1, j\neq o}^{n} \bar{\lambda}_j y_{rj} > 0$, $\sum_{j=1, j\neq o}^{n} \bar{\lambda}_j x_{ij} < 0$ and $\sum_{j=1, j\neq o}^{n} \bar{\lambda}_j = 1$. We define the new output vector as: $\tilde{y}_r = \min\{y_{ro}, \sum_{j=1, j\neq o}^{n} \bar{\lambda}_j y_{rj}\}$ , $r = 1, \dots, s$ . So we have two different cases. In the first case, $r \in O$ so $y_{ro} > 0$. Since $\tilde{y}_r \leq y_{ro}$ , that is $0 < \tilde{y}_r \leq y_{ro}$ , $r \in O$ , then the constraint related to the outputs $r \in O$ is satisfied. If $r \in \bar{O}$, then $y_{ro} < 0$ , i.e., $y_{ro} = \tilde{y}_r$, so $\tilde{y}_r \leq y_{ro} < 0$ , which means that the constraint related to the output $r \in \bar{O}$ is also satisfied.

We define the new input vector as: $\tilde{x}_i = \max\{x_{io}, \sum_{j=1, j\neq o}^{n} \bar{\lambda}_j x_{ij}\}$ , $i = 1, \dots, m$ .

Therefore $x_{io} \leq \tilde{x}_i$ and $\tilde{x}_i \leq \sum_{j=1, j\neq o}^{n} \bar{\lambda}_j x_{ij}$ . We have two different cases; in the first case $i \in I$, so $x_{io} > 0$ . In this case, $x_{io} = \tilde{x}_i$ , and hence $0 < x_{io} \leq \tilde{x}_i,\ i \in I$. Therefore, the constraints related to the input $i \in I$ are satisfied. In the second case, assume $i \in \bar{I}$ , then $x_{io} < 0$. Therefore, $\tilde{x}_i < 0$ and hence $x_{io} \leq \tilde{x}_i < 0$ So, the constraints related to the inputs $i \in \bar{I}$ are also satisfied. Thus $(\tilde{x}, \tilde{y})$ is a feasible solution for Model (3), and the proof is complete.

***Theorem 4:***

The optimal values of the objective function of Model (3) in evaluating $DMU_o$ are finite, and they are greater than or equal to one.

**Proof.** First, we show that the objective function values in the case of feasibility of Model (3) are greater than or equal to one. Considering the constraints of the problem, we have $\frac{\bar{x}_{io}}{x_{io}} \geq 1 \quad i \in I$ , $\frac{\bar{x}_{io}}{x_{io}} \leq 1 \quad i \in \bar{I}$ , $\frac{\bar{y}_{ro}}{y_{ro}} \geq 1 \quad r \in \bar{O}$ , $\frac{\bar{y}_{ro}}{y_{ro}} \leq 1 \quad r \in O$ .

Now, we show that the objective function value will be greater than or equal to one.

This is true when we have at least one negative output or one positive input in the numerator of the objective function and also at least one positive output or one negative input in the denominator of the objective function. Considering $\frac{\bar{x}_i}{x_{io}} \geq 1$ , $i \in I$ , we have $\sum_{i\in I} \frac{\bar{x}_i}{x_{io}} \geq m_1$ . Moreover, $\frac{\bar{y}_r}{y_{ro}} \geq 1 \quad r \in \bar{O}$ so $\sum_{r\in\bar{O}} \frac{\bar{y}_r}{y_{ro}} \geq s - s_1$ . Similarly, as $\frac{\bar{x}_i}{x_{io}} \leq 1, i \in \bar{I}$ and $\frac{\bar{y}_r}{y_{ro}} \leq 1$ , $r \in O$ then $\sum_{i\in\bar{I}} \frac{\bar{x}_i}{x_{io}} \leq m - m_1$ and $\sum_{r\in O} \frac{\bar{y}_r}{y_{ro}} \leq s_1$ .

This means that $\frac{1}{m_1+s-s_1}\left(\sum_{i\in I} \frac{\bar{x}_i}{x_{io}} + \sum_{r\in\bar{O}} \frac{\bar{y}_r}{y_{ro}}\right) \geq 1$ and $\frac{1}{m-m_1+s_1}\left(\sum_{i\in\bar{I}} \frac{\bar{x}_i}{x_{io}} + \sum_{r\in O} \frac{\bar{y}_r}{y_{ro}}\right) \leq 1$ , Therefore $\delta^* \geq 1$ .

Now, we show that the objective function value is finite in the case of feasibility of Model (3). We know that the optimal value of Model (3) and that of the linear Model (4) are equal. So, we can show that the optimal value of the linear Model (4) is finite. To this end, we show that the dual problem corresponding to the linear model (4) is always feasible. The dual of the linear model (4) is:

$$\begin{aligned}
Max\quad & (m-m_1+s_1)u_0 \\
s.t\quad & v_0+\sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\le 0 \qquad j=1,\dots,n,\quad j\ne o, \\
& \frac{u_0}{x_{io}}+v_i+w_i\ge 0, \qquad i\in \bar{I}, \\
& v_i+w_i=\frac{1}{(m_1+s-s_1)x_{io}} \qquad i\in I, \\
& \frac{u_0}{y_{ro}}-u_r-\gamma_r\le 0, \qquad r\in O, \\
& -u_r-\gamma_r\ge\frac{1}{(m_1+s-s_1)y_{ro}} \qquad r\in \bar{O}, \\
& -v_0+\sum_{r=1}^{s}\gamma_r y_{ro}-\sum_{i=1}^{m}w_i x_{io}+\sum_{i\in I}\alpha_i x_{io}\le 0, \\
& v_i\ge 0,\quad w_i\ge 0 \quad i=1,\dots,m, \\
& u_r\ge 0,\quad \gamma_r\ge 0 \quad r=1,\dots,s,\quad \alpha_i\ge 0 \quad i\in I.
\end{aligned} \tag{7}$$

We set $u_r=\gamma_r=0 \quad r=1,\dots,s$ , $v_i=0 \quad i=1,\dots,m$ , $\alpha_i=0 \quad i\in I$ , $w_i=0 \quad i\in\bar{I}$ , $w_i=\frac{1}{x_{io}(m_1+s-s_1)} \quad i\in I$ and $u_0=\min\{x_{io}\,|\,i\in\bar{I}\}$ .

So, the above model is always feasible and thus the linear problem has a finite optimal value and the proof is complete.

Regarding the constraints of Model (3), if $DMU_o$ is inefficient or is a non-extreme efficient DMU, the optimal value of Model (3) will be equal to one. Because in the optimal value corresponding to these DMUs, we have $\bar{x}_i^*=x_{io},\ i=1,\dots,m$ and $\bar{y}_r^*=y_{ro},\ r=1,\dots,s$ . So, the objective function value is equal to one, since the PPS frontier does not change when such DMUs are excluded.

***Input/output Oriented Super-efficiency:***

The input-oriented model corresponding to Model (3) is presented as follows.

$$\begin{aligned}
[SuperSBMNI]\quad \delta_I^*=Min\quad & \frac{\frac{1}{m_1}\sum_{i\in I}\bar{x}_i/x_{io}}{\frac{1}{m-m_1}\sum_{i\in\bar{I}}\bar{x}_i/x_{io}} \\
s.t\quad & \sum_{j=1,j\ne o}^{n}\lambda_j x_{ij}\le\bar{x}_i, \qquad i=1,\dots,m, \\
& \sum_{j=1,j\ne o}^{n}\lambda_j y_{rj}\ge\bar{y}_r, \qquad r=1,\dots,s, \\
& \sum_{j=1,j\ne o}^{n}\lambda_j=1,\quad \lambda_j\ge 0, \qquad j=1,\dots,n, \\
& 0<x_{io}\le\bar{x}_i \quad i\in I,\quad x_{io}\le\bar{x}_i<0 \quad i\in\bar{I}. \\
& 0<\bar{y}_r=y_{ro} \quad r\in O,\quad \bar{y}_r=y_{ro}<0 \quad r\in\bar{O}.
\end{aligned} \tag{8}$$

***Theorem 5:***

For the enhanced input $\tilde{x}_{io}=x_{io}+\Delta x_i<0 \quad i\,(\Delta x_i\ge 0)$ and $\tilde{x}_{io}=x_{io}-\Delta x_i>0 \quad (\Delta x_i\ge 0)$ the optimal objective function value $\delta_I^*$ corresponding to this change satisfies the following relation.
$\delta_I^*\le\delta_I^*(\Delta x)$ .

**Proof.** The linear programming for this perturbed problem problem is express as follows.

$$\begin{aligned}
& [SuperSBMNI(\Delta x)]\qquad \delta_I^*(\Delta x)=Min\quad \frac{\frac{1}{m_1}\sum_{i\in I}\frac{x_i}{x_{io}-\Delta x_i}}{\frac{1}{m-m_1}\sum_{i\in\bar{I}}\frac{\bar{x}_i}{x_{io}+\Delta x_i}} \\
& s.t\quad \sum_{j=1,j\ne o}^{n}\lambda_j x_{ij}\le\bar{x}_i, \qquad i=1,\dots,m, \\
& \sum_{j=1,j\ne o}^{n}\lambda_j y_{rj}\ge\bar{y}_r, \qquad r=1,\dots,s, \\
& \sum_{j=1,j\ne o}^{n}\lambda_j=1,\ \lambda_j\ge 0, \qquad j=1,\dots,n,
\end{aligned} \tag{9}$$

$0 < x_{io} - \Delta x_i \leq \bar{x}_i \quad i \in I, x_{io} + \Delta x_i \leq \bar{x}_i < 0 \quad i \in \bar{I}$

$0 < \bar{y}_r = y_{ro} \quad r \in O, \quad \bar{y}_r = y_{ro} < 0 \quad r \in \bar{O}.$

For any optimal solution $(\bar{x}^*, \bar{y}^*)$ of the above chaotic problem, $\bar{x}_i^* + \Delta x_i \quad i \in I$ , $\bar{x}_i^* - \Delta x_i \quad i \in \bar{I}$ $\bar{y}_r = y_{ro} \quad r = 1, \ldots, s$ , will be a feasible solution for the Model (8)(Super SBMNI problem), so

$$\delta_I^* \leq \frac{\frac{1}{m_1}\sum_{i\in I}\frac{\bar{x}_i^*+\Delta x_i}{x_{io}}}{\frac{1}{m-m_1}\sum_{i\in \bar{I}}\frac{\bar{x}_i^*-\Delta x_i}{x_{io}}} \leq \frac{\frac{1}{m_1}\sum_{i\in I}\frac{\bar{x}_i^*}{x_{io}-\Delta x_i}}{\frac{1}{m-m_1}\sum_{i\in \bar{I}}\frac{\bar{x}_i^*}{x_{io}+\Delta x_i}} = \delta_I^*(\Delta x)$$

The above inequality holds since, if $i \in I$ we have $0 < x_{io} - \Delta x_i \leq \bar{x}_i$ then $\frac{\bar{x}_i^*}{x_{io}-\Delta x_i} - \frac{\bar{x}_i^*+\Delta x_i}{x_{io}} = \frac{\Delta x_i(\bar{x}_i^* - x_{io} + \Delta x_i)}{x_{io}(x_{io}-\Delta x_i)} \geq 0$ and if $i \in \bar{I}$ we have $x_{io} + \Delta x_i \leq \bar{x}_i < 0$ , then $\frac{\bar{x}_i^*-\Delta x_i}{x_{io}} - \frac{\bar{x}_i^*}{x_{io}+\Delta x_i} = \frac{\Delta x_i(\bar{x}_i^* - x_{io} - \Delta x_i)}{x_{io}(x_{io}+\Delta x_i)} \geq 0$ , and the proof is complete.

***Theorem 6:*** $\delta^* \leq \delta_I^*$ , $\delta^* \leq \delta_O^*$ .

**Proof.** The proof is obvious considering the fact that the feasible regions of the input-oriented and output-oriented models are subsets of the feasible region of Model (3). The output-oriented model is as follows.

$$\begin{aligned} [SuperSBMNO] \quad \delta_O^* = Min \quad & \frac{\frac{1}{s-s_1}\sum_{r\in \bar{O}}\frac{y_r}{y_{ro}}}{\frac{1}{s_1}\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}}} \\ s.t \quad & \sum_{j=1,j\neq o}^{n} \lambda_j x_{ij} \leq \bar{x}_i, \quad i = 1, \ldots, m, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j y_{rj} \geq \bar{y}_r, \quad r = 1, \ldots, s, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j = 1, \ldots, n, \\ & 0 < y_{ro} \leq \bar{y}_r \quad r \in O, \quad y_{ro} \leq \bar{y}_r < 0 \quad r \in \bar{O}, \\ & 0 < \bar{x}_i = x_{io} \quad i \in I, \quad \bar{x}_i = x_{io} < 0 \quad i \in \bar{I}. \end{aligned} \tag{10}$$

***Numerical Example 1:***

we elaborate on the slack-base super efficiency problem by a numerical example. To this end, we make use of the data sets employed by Emrouznejad *et al*. (2010). Table 1 shows data for 10 DMU with activity vector $(x, y, z)$ . Each DMU uses input $x$ to produce outputs $y$, $z$. The output $z$ is always positive. The output $y$ is positive for some DMUs and negative for others. To obtain the efficiency of each DMU, we use the following model (SORM) that proposed by Emrouznejad *et al*. (2010). They introduce in respect of variable $y$ two variables: $y^1$ and $y^2$ as follows:

$y^1 = y$ and $y^2 = 0$ ; if $y \geq 0$ . $y^1 = 0$ and $y^2 = y$ ; if $y < 0$ .

The a mount of efficiency is calculated as $\frac{1}{h^*}$ .

$$\begin{aligned} h^* = Max \quad & h \\ s.t \quad & \sum_{j=1}^{n} \lambda_j x_j \leq x_o \\ & \sum_{j=1}^{n} \lambda_j z_j \geq h z_o \\ & \sum_{j=1}^{n} \lambda_j y_j^1 \geq h y_o^1 \\ & \sum_{j=1}^{n} \lambda_j y_j^2 \leq h y_o^2 \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \quad j = 1, \ldots, n. \end{aligned} \tag{11}$$

**Table 1:** Input-output Data for 10 DMUs.

| DMU | (x) (input) | (y) (output) | (z) (output) | DMU | (x) (input) | (y) (output) | (z) (output) |
|---|---|---|---|---|---|---|---|
| DMU1 | 12 | 15 | 11 | DMU6 | 50 | -8 | 27 |
| DMU2 | 35 | 18 | 6 | DMU7 | 35 | -18 | 6 |
| DMU3 | 25 | 20 | 13 | DMU8 | 40 | -10 | 22 |
| DMU4 | 22 | 12 | 20 | DMU9 | 25 | -7 | 19 |
| DMU5 | 40 | -10 | 25 | DMU10 | 16 | 26 | 8 |

The results of the calculations are provided in Table 2. It can be observed that this model introduces DMUs 1, 4, 5, 6, and 10 as efficient DMUs, but does not deal with their efficiency ranking. In order to rank the DMUs and obtain the amount of super-efficiency corresponding to each one, Model (3) can be employed. Table (2) contains super-efficiency value $\delta^*$ , efficiency ranking, projection points, input and output slacks, and the reference set. It can be seen that DMU 1 has the highest rank and DMUs 10, 4, 6, 5 hold the next places, respectively. $\bar{x}^*$ , $\bar{y}^*$ , $\bar{z}^*$ , indicate the projection points corresponding to each DMU on the new PPS frontier after the exclusion of the DMU under evaluation. The input and output slacks corresponding to each DMU are included in the Table, as well. In the last column, the reference set and the projection multipliers for each DMU are provided. By solving Model (3), the efficiency ranking of extreme efficient DMUs is obtained. For an inefficient or non-extreme efficient DMU, the corresponding super-efficiency value of 1 will be obtained.

**Table 2:** Results of Super SBMN model.

| DMU | SBMN | SORM | Rank | Projected point (input-output slacks) | | | Reference | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\delta^*$ | $\frac{1}{h^*}$ | | $\bar{x}^*$ | $\bar{y}^*$ | $\bar{z}^*$ | Referent | | |
| | | | | $(s^{-*})$ | $(s_1^{+*})$ | $(s_2^{+*})$ | $(\lambda^*)$ | | |
| DMU1 | 1.4583 | 1 | 1 | 17.5 | 15 | 11 | D4 | D10 | |
| | | | | (0) | (7.5) | (0) | (0.25) | (0.75) | |
| DMU2 | 1 | 0.7077 | - | 35 | 18 | 6 | D10 | | |
| | | | | (19) | (8) | (2) | (1) | | |
| DMU3 | 1 | 0.9953 | - | 25 | 20 | 13 | D4 | D10 | |
| | | | | (6.4286) | (0) | (0.1429) | (0.4286) | (0.5714) | |
| DMU4 | 1.163 | 1 | 3 | 22 | 12 | 14.3935 | D1 | D3 | D5 |
| | | | | (0) | (0) | (0) | (0.4516) | (0.357) | (0.1914) |
| DMU5 | 1.0204 | 1 | 5 | 40 | -10 | 24.5 | D4 | D6 | |
| | | | | (0) | (9.1429) | (0) | (0.3571) | (0.6429) | |
| DMU6 | 1.1 | 1 | 4 | 50 | -8 | 24.5455 | D4 | D5 | |
| | | | | (11.6363) | (0) | (0) | (0.0909) | (0.9091) | |
| DMU7 | 1 | 0.2541 | - | 35 | -18 | 6 | D6 | D10 | |
| | | | | (0) | (42.0659) | (3.0808) | (0.0569) | (0.9431) | |
| DMU8 | 1 | 0.8803 | - | 40 | -10 | 22 | D5 | D6 | D9 |
| | | | | (0) | (1.5738) | (0) | (0.4672) | (0.0246) | (0.5082) |
| DMU9 | 1 | 0.9121 | - | 25 | -7 | 19 | D4 | D10 | |
| | | | | (3.5) | (20.1667) | (0) | (0.9167) | (0.0833) | |
| DMU10 | 1.2224 | 1 | 2 | 16 | 16.5385 | 8 | D1 | D3 | |
| | | | | (0) | (0) | (3.6154) | (0.6923) | ( 0.3077) | |

***Extension:***

Model (3) can be useful when, first, the input and output data are positive or negative (non-zero) and, second, in the objective function we have $m - m_1 + s_1 \neq 0$ and $s - s_1 + m_1 \neq 0$ . In the case where these two assumptions are not satisfied, we can do the following.

***1. Zero in Input and Output Data:***

If some of the inputs or outputs of $DMU_o$ are zero, similar to the method presented in Tone (2002), we can do the following.

First, consider the case in which some input components are zero. we have the following cases:

Case 1) $DMU_o$ has no function as to the input 1. In this case, the variable $\bar{x}_1$ in Model (3) is set equal to zero, the expression $\frac{\bar{x}_1}{x_{1o}}$ in the objective function is assigned a value of one, and the expression corresponding to $\bar{x}_1$ in the constraints is eliminated.

Case 2) $DMU_o$ has the function 1 but incidentally its observed value is zero.

Here, if $x_{ij} \geq 0$ , $j = 1, \dots, n$ , we set the value of $x_{io}$ to an infinitesimal, e.g., $e$ =(The smallest positive $x_{ij}$

value in the data set with positive input)/100, and if $x_{ij} \le 0$ , $j = 1, \dots, n$ , we set it to an infinitesimal, e.g., *e*=(The largest negative $x_{ij}$ value in the data set with negative input)/100. Else we put $\varepsilon = (\varphi)$ /100 such that

$\varphi = \min\{x_{ij} \quad i \in I, |x_{ij}| \quad i \in \bar{I}\}$ .

For the case where we have zero output components, a procedure similar to that for inputs is followed.

***Numerical Example 2:***

Table 3 shows the data set from the notional effluent processing system extracted by Sharp *et al*. (2006). As can be observed, there are 13 DMUs each with one positive input (cost), one non-positive input (effluent), one positive output (saleable), and two non-positive outputs (Methane and CO2).

**Table 3:** Notional effluent processing system.

| DMU | $(I_1)$ Cost | $(I_2)$ Effluent | $(O_1)$ Saleable | $(O_2)CO_2$ | $(O_3)$ Methane |
|---|---|---|---|---|---|
| DMU1 | 1.03 | -.05 | 0.56 | -0.09 | -0.44 |
| DMU2 | 1.75 | -0.17 | 0.74 | -0.24 | -0.31 |
| DMU3 | 1.44 | -0.56 | 1.37 | -0.35 | -0.21 |
| DMU4 | 10.8 | -0.22 | 5.61 | -0.98 | -3.79 |
| DMU5 | 1.3 | -0.07 | 0.49 | -1.08 | -0.34 |
| DMU6 | 1.98 | -0.1 | 1.61 | -0.44 | -0.34 |
| DMU7 | 0.97 | -0.17 | 0.82 | -0.08 | -.043 |
| DMU8 | 9.82 | -2.32 | 5.61 | -1.42 | -1.94 |
| DMU9 | 1.59 | 0 | 0.52 | 0 | -0.37 |
| DMU10 | 5.96 | -0.15 | 2.14 | -0.52 | -0.18 |
| DMU11 | 1.29 | -0.11 | 0.57 | 0 | -0.24 |
| DMU12 | 2.38 | -0.25 | 0.57 | -0.67 | -0.43 |
| DMU13 | 10.3 | -0.16 | 9.56 | -0.58 | 0 |

It can be seen that DMU 9 has the second input and the second output equal to zero. As was stated earlier, we can assign -0.0005 and 0.0008 to the second input and output of this DMU, respectively; DMU 11 a second output of zero, to which we assign -0.0008; and DMU 13 has a third output of zero, to which we assign -0.0018. To calculate the amounts of SBMN values for these DMUs, we employ Model (3). In doing so, we substitute the above-mentioned values for the zero components and solve Model (3). Table(4) contains the results of these calculations, including the amounts of Super SBMN value (SBMN), approach score of Sharp *et al*. (2006)(Modified slacks based measure (MSBM)), efficiency rank.

**Table 4:** Input-output data for 10 DMUs.

| DMU | SBMN $\delta^*$ | Rank Rank | MSBM $\rho^*$ | DMU | SBMN $\delta^*$ | Rank Rank | MSBM $\rho^*$ |
|---|---|---|---|---|---|---|---|
| DMU1 | 1 | - | 0.88 | DMU7 | 1.1452 | 5 | 1 |
| DMU2 | 1 | - | 0.74 | DMU8 | 1.7359 | 4 | 1 |
| DMU3 | 1.9252 | 3 | 1 | DMU9 | 1 | - | 0.89 |
| DMU4 | 1 | - | 0.56 | DMU10 | 1 | - | 0.72 |
| DMU5 | 1 | - | 0.7 | DMU11 | 2.7444 | 2 | 1 |
| DMU6 | 1 | - | 0.78 | DMU12 | 1 | - | 0.68 |
| DMU13 | 55.8735 | 1 | 1 | | | | |

**Table 5:** Results of Super SBMN model.

| DMU | Projected point(slacks) $\bar{x}_1^*$ $(s_1^{-*})$ | $\bar{x}_2^*$ $(s_2^{-*})$ | $\bar{y}_1^*$ $(s_1^{+*})$ | $\bar{y}_2^*$ $(s_2^{+*})$ | $\bar{y}_3^*$ $(s_3^{+*})$ | Reference Referent $(\lambda^*)$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| DMU1 | 1.03 | -0.05 | 0.56 | -0.09 | -0.44 | D7 | D11 | D12 | | |
| | (0) | (0.1179) | (0.2357) | (0) | (0.0235) | (0.9026) | (0.0709) | (0.0265) | | |
| DMU2 | 1.75 | -0.17 | 0.74 | -0.24 | -0.31 | D3 | D6 | D8 | D10 | D11 |
| | (0) | (0) | (0.4228) | (0) | (0) | (0.0623) | (0.4403) | (0.0164) | (0.4795) | (0.0014) |
| DMU3 | 2.644 | -0.4608 | 1.37 | -0.35 | -0.5098 | D8 | D11 | | | |
| | (0) | (0) | (0) | (0.1246) | (0) | (0.1587) | (0.8413) | | | |
| DMU4 | 10.8 | -0.22 | 5.61 | -0.98 | -3.79 | D8 | D11 | D13 | | |
| | (2.3785) | (1.2228) | (0) | (0) | (2.5871) | (0.598) | (0.1766) | (0.2254) | | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| DMU5 | 1.3 | -0.07 | 0.49 | -1.08 | -0.34 | D9 | D11 | | |
| | (0) | (0.03634) | (0.0783) | (1.0792) | (0.0957) | (0.0333) | (0.9667) | | |
| DMU6 | 1.987 | -0.1 | 1.61 | -0.44 | -0.34 | D3 | D5 | D8 | |
| | (0) | (0.5625) | (0.015) | (0) | (0.0128) | (0.9073) | (0.027) | (0.0657) | |
| DMU7 | 1.3925 | -0.17 | 0.82 | -0.08 | -0.43 | D1 | D3 | D11 | D13 |
| | (0) | (0) | (0) | (0) | (0.1612) | (0.1843) | (0.1563) | (0.6453) | (0.0141) |
| DMU8 | 9.82 | -0.3529 | 5.61 | -1.42 | -1.94 | D3 | D13 | | |
| | (3.7931) | (0) | (0) | (0.9509) | (1.8387) | (0.4823) | (0.5177) | | |
| DMU9 | 1.59 | -0.0005 | 0.52 | -0.0008 | -0.37 | D11 | | | |
| | (0.3) | (0.1095) | (0.05) | (0) | (0.13) | (1) | | | |
| DMU10 | 5.96 | -0.15 | 2.14 | -0.52 | -0.18 | D3 | D6 | D12 | D13 |
| | (0) | (0) | (3.2005) | (0.0026) | (0) | (0.0279) | (0.4379) | (0.0567) | (0.4775) |
| DMU11 | 1.59 | -5.0013 | 0.52 | -7.9993 | -0.37 | D9 | | | |
| | (0) | (0) | (0) | (0) | (0) | (1) | | | |
| DMU12 | 2.38 | -0.25 | 0.57 | -0.67 | -0.43 | D5 | D6 | D8 | D10 | D11 |
| | (0) | (0) | (0.7973) | (0) | (0) | (0.3499) | (0.3797) | (0.0705) | (0.0477) | (0.1522) |
| DMU13 | 10.3087 | -0.16 | 2.1212 | -0.58 | -0.1807 | D3 | D10 | | |
| | (4.4502) | (0) | (0) | (0.0641) | (0) | (0.0244) | (0.9756) | | |

Table 5 contains the results of these calculations, including the amounts of input and output vector projections, input and output slacks, the reference set, and the projection multipliers corresponding to the DMU under assessment. By using the model proposed by Sharp *et al*. (2006) to evaluate the efficiency of units, DMUs 3, 7, 8, 11, and 13 are evaluated as efficient. To rank these units, we use Model (3), by which DMU 13 has the highest rank and DMUs 11, 3, 8, 7 hold the next places.

2. $m - m_1 + s_1 = 0$ or $s - s_1 + m_1 = 0$ .

When $s - s_1 + m_1 = 0$ , i.e., we do not have at least one positive input or one negative output in the numerator of the objective function , we can use the following objective function instead of the objective function of Model (3).

$$\frac{1}{\frac{1}{m-m_1+s_1}\left(\sum_{i\in\bar{I}}\frac{x_i}{x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}}\right)}$$

When $m - m_1 + s_1 = 0$ , i.e., there is not at least one negative input or one positive output in the denominator of the objective function , we can use the following objective function in place of the objective function of Model (3), i.e., we set the denominator equal to one. $\frac{1}{m_1+s-s_1}\left(\sum_{i\in I}\frac{\bar{x}_i}{x_{io}}+\sum_{r\in\bar{O}}\frac{\bar{y}_r}{y_{ro}}\right)$ .

***Numerical Example 3:***

Table 6 shows the data for ten hypothetical DMUs each with one input and one output. The input and output vectors contain positive and negative values. We can solve Model (3) too obtain the super-efficiency amount corresponding to each DMU.

**Table 6:**: Input-output Data for 10 DMUs.

| DMU | (x) (input) | (y) (output) | DMU | (x) (input) | (y) (output) |
|---|---|---|---|---|---|
| DMU1 | -8 | 2 | DMU6 | 6 | 9 |
| DMU2 | -8 | 3 | DMU7 | 8 | 9 |
| DMU3 | -6 | 5 | DMU8 | -3 | -3 |
| DMU4 | -4 | 6 | DMU9 | -4 | 5 |
| DMU5 | 1 | 7.5 | DMU10 | 6 | -2 |

For solving model (3) corresponding to DMUs 1, 2, 3, 4, 9, since $s - s_1 + m_1 = 0$, we can use the proposed objective function $\frac{1}{\frac{1}{m-m_1+s_1}\left(\sum_{i\in\bar{I}}\frac{x_i}{x_{io}}+\sum_{r\in O}\frac{\bar{y}_r}{y_{ro}}\right)}$ .

To solve Model (3) for DMU 10, since $m - m_1 + s_1 = 0$ we can employ the proposed objective function $\frac{1}{m_1+s-s_1}\left(\sum_{i\in I}\frac{\bar{x}_i}{\alpha x_{io}} + \sum_{r\in\bar{O}}\frac{\bar{y}_r}{y_{ro}}\right)$ .

Table 7 provides the , input and output slacks, the reference set, and the projection multipliers corresponding to the DMU under evaluation. As can be observed, DMUs 1, 2, 3, 4, 6, and 7 are efficient, among which DMUs 2, 3, 4, and 6 are extreme efficient. Among the latter group, DMU 3 has the highest rank and DMUs 6, 2, and 4 hold the next places. Note that inefficient and non-extreme efficient DMUs have an Super SBMN value (SBMN) of 1.

**Table 7:** Results of Super SBMN model.

| DMU | SBMN | Rank | Projected point | | Slacks | | Reference | |
|---|---|---|---|---|---|---|---|---|
| | $\delta^*$ | Rank | $\bar{x}^*$ | $\bar{y}^*$ | ($s_1^{-*}$) | ($s_1^{+*}$) | Referent ($\lambda^*$) | |
| DMU1 | 1 | - | -8 | 2 | 0 | 1 | D2(1) | |
| DMU2 | 1.0435 | 3 | -7.333 | 3 | 0 | 0 | D3(1/3) | D1(2/3) |
| DMU3 | 1.0526 | 1 | -6 | 4.5 | 0 | 0 | D2(1/2) | D4(1/2) |
| DMU4 | 1.0244 | 4 | -4 | 5.7143 | 0 | 0 | D3(0.7143) | D5(0.2857) |
| DMU5 | 1 | - | 1 | 7.5 | 0 | 0 | D4(1/2) | D6(1/2) |
| DMU6 | 1.05 | 2 | 6 | 8.5715 | 0 | 0 | D5(0.2857) | D7(0.7143) |
| DMU7 | 1 | - | 8 | 9 | 2 | 0 | D6(1) | |
| DMU8 | 1 | - | -3 | -3 | 0 | 7.1875 | D1(0.6875) | D7(0.3125) |
| DMU9 | 1 | - | -4 | 5 | 0 | 1 | D4(1) | |
| DMU10 | 1 | - | 6 | -2 | 8.0833 | 0 | D7(0.0833) | D8(0.9167) |

***Conclusion:***

Standard DEA models cannot be employed with negative data. In recent years, some models have been proposed to deal with negative data. However, all these models evaluate a number of DMUs as efficient and assign to them an efficiency amount of unity, but make no mention of the priority of one unit over another and do not rank the DMUs. In the present paper, a super-efficiency measure based on input and output slacks was proposed. In the paper, numerical examples were used to elaborate on the theorems and the feasibility conditions of the proposed model. We showed that an efficiency rank can be provided for each extreme efficient unit by using the proposed model to compare these DMUs. In doing so, the distance of the DMU under evaluation from the new PPS produced by the exclusion of $DMU_o$ and considering the constraints $\bar{x} \geq x_o$ and $\bar{y} \leq y_o$ is obtained by norm-1. The norm-2 and Chebychev's norm can also be used for measuring this distance. The dual of the proposed model was provided in the present paper, as well. The above discussions can be made by the dual model, too. Also, the model can be extended to imprecise data.

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