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The Simple proof of Legendre's Condition in Economics

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Abstract: The proof of Legendre's condition, to some extent is not understandable in different mathematics and mathematical economics textbooks. This article tries to present another proof for this condition by using reductio ad absurdum and sign determination of quadratic equation in two simple ways.

Key words: Legendre's Condition, Reductio ad Absurdum, Sign determination of quadratic equation.

INTRODUCTION

Calculus of variations has a lot of applications in different sciences. Considering the importance of calculus of variations as the oldest and primary solution in control problem i.e. selecting time path, from the initial point to the final point in order to maximize or minimize the amount of given integral function. The first-order necessary condition in calculus of variations is called Euler's equation. The other necessary condition in which the classical problem solution of calculus of variations must be satisfied is called the second-order necessary condition or Legendre's condition.
The classical problem of calculus variation is:

(1) Max or min $V[y] = \int_0^T F[t, y(t), y'(t)]\,dt$

s.t $y(0) = A, y(T) = Z$ (A, T, Z are given).

1-1-The First-order Necessary Condition:

The most important condition of first-order is Euler's equation. Considering Euler's equation proof, it can be yielded from the following relation:

$$(2)\quad \frac{dV}{d\varepsilon}_{(\varepsilon=0)} = 0$$

For all $t \in [0, T]$ Euler's equation is:

$$(3)\quad F_y - \frac{d}{dt}F_{y'} = 0$$

1-2-Legendre's Necessary Condition:

The condition which is similar to static mood of second-order necessary condition in classical planning problem is called Legendre's condition.(Douglas, Jesse, 2008) The Legendre's necessary condition for maximization and minimization yields from the following relations:

$$(4)\quad \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} \le 0 \ for\ maximum, \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} \ge 0 \ for\ minimum$$

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Big advantage of Legendre's condition is that it is very easy to calculate. Because this condition is nothing but sign of $F_{y'y'}$. Legendre as a mathematician once thought that he has discovered a good enough condition but his thoughts were wrong. In fact, unequal weak form of this condition is a correct necessary condition:(Chiang, C. Alfa, 1992)

(5) Maximization $V[y] \quad \rightarrow F_{y'y'} \leq 0$ for all $t \in [0,T]$

(6) Minimization $V[y] \quad \rightarrow F_{y'y'} \geq 0$ for all $t \in [0,T]$

[Legendre's necessary condition]

The derivative of $F_{y'y'}$ is calculated in extremum.
The second-order derivative is equal to:

$$(7) \quad \frac{d^2V}{d\varepsilon^2} = \frac{d}{d\varepsilon}\left(\frac{dV}{d\varepsilon}\right) = \frac{d}{d\varepsilon}\int_0^T \left[F_y p(t) + F_{y'} p'(t)\right] dt$$

The second-order derivative will be appeared after simplification as follows:

$$(8) \quad \frac{d^2V}{d\varepsilon^2} = \int_0^T \left[F_{yy} p^2(t) + 2F_{yy'} p(t) p'(t) + F_{y'y'} p'^2(t)\right] dt$$

Where p is a disturbance curve which is selected arbitrarily to yield Euler's equation.

1-3-Proof Method:

Reductio ad absurdum and sign determination of quadratic equation has been used for proofing Legendre's necessary condition in this article. Since this article has tried to prove Legendre's necessary condition for maximization, it can be also generalized for minimization process.

2- The Simple Proof of Legendre's Condition:

2-1-The First Method:

Considering the second derivative of *V* with respect to ε, for Legendre's necessary condition in maximum we will have:

$$(9) \quad \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} \leq 0 \ for \ \ maximum \rightarrow \quad F_{y'y'} \leq 0$$

By using reductio ad absurdum we suppose $F_{y'y'}$ is positive, so we have:

$$(10) \quad F_{y'y'} > 0$$

We are going to consider circumstances in which $\frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)}$ is positive and shows a contradiction, so we have:

$$(11) \quad \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} > 0$$

Proof

Since $F_{y'y'}$ is positive, by using this assumption between *F* derivatives confirm following relation:

$$(12)\quad F_{yy'}^2 - F_{yy}F_{y'y'} < 0$$

Therefore it can be concluded that the following quadratic equation is positive:

$$(13)\quad F_{yy} + 2F_{yy'}U + F_{y'y'}U^2 > 0$$

Now suppose that $U = \frac{p(t)'}{p(t)}$, so by substitution in (13) and multiply it in $p^2(t)$ we have:

$$(14)\quad F_{yy}p^2(t) + 2F_{yy'}p(t)p'(t) + F_{y'y'}p'^2(t) > 0$$

If we integrate from (14) in the intervals between $[o,T]$, so we have:

$$(15)\quad \int_0^T \left[F_{yy}p^2(t) + 2F_{yy'}p(t)p'(t) + F_{y'y'}p'^2(t)\right]dt > 0$$

Thus, the statement of (15) is $\frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)}$ which is greater than zero, and we have:

$$(16)\quad \frac{d^2V}{d\varepsilon^2} = \int_0^T \left[F_{yy}p^2(t) + 2F_{yy'}p(t)p'(t) + F_{y'y'}p'^2(t)\right]dt > 0$$

The statement of (16) is the same contradiction which we have required.
As a result, we have:

$$(17)\quad \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} \leq 0 \ for \ \ maximum \rightarrow \quad F_{y'y'} \leq 0$$

This proof also can be generalized for minimization.

2-2- The Second Method:

We suppose that $g(t) \equiv F_{y'y'}$ and $f(t) \equiv F_{yy}$ and $h(t) \equiv F_{yy'}$.

Now by using reductio ad absurdum, we suppose that there is a point such as t_0 in the interval of $(0,T)$, where:

$$(18)\quad g(t_0) > 0$$

Now we can consider *p* in a way that we have:

$$(19)\quad \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} > 0$$

This is a contradiction.

proof

Since g is a continuous function, $\delta > 0$ can be selected as a very small number so that:

$$(20) \quad g(t) > 0, \quad \forall t \in [t_0 - \delta, t_0 + \delta] \subset (0, T)$$

Furthermore, it should be noticed that *g(t)* can be positive in other distances but it is surely positive in this interval because *g(t)* function is continuous. Because of simplification, we consider proof in this distance. Now we consider a specific p function which had necessary conditions for such functions.

$$(21) \quad p(t) = \begin{cases} p(t) & t \in (t_0 - \delta, t_0 + \delta) \\ o & \text{otherwise} \end{cases}$$

[For example, following function and it's derivative]

$$(22) \quad \left[p(t) = \begin{cases} \sin^2 \dfrac{\pi(t - t_0)}{\delta} & t \in (t_0 - \delta, t_0 + \delta) \\ o & \text{otherwise} \end{cases} \qquad p'(t) = \frac{\pi}{b} \sin 2\left(\frac{\pi(t - t_0)}{\delta}\right) \right]$$

Now as for $g(t) \equiv F_{y'y'}$ in open interval of $(t_0 - \delta, t_0 + \delta)$ is positive due to in closed interval of $[t_0 - \delta, t_0 + \delta]$ is positive and considering given assumption, we have in this interval:

$$(23) \quad F_{yy'}^2 - F_{yy} F_{y'y'} < 0$$

As before, we can conclude in distance of $(t_0 - \delta, t_0 + \delta)$:

$$(25) \quad \frac{d^2V}{d\varepsilon^2} = \int_0^T \left[F_{yy} p^2(t) + 2F_{yy'} p(t) p'(t) + F_{y'y'} p'^2(t) \right] dt$$
$$= \int_{t_0 - \delta}^{t_0 + \delta} \left[F_{yy} p^2(t) + 2F_{yy'} p(t) p'(t) + F_{y'y'} p'^2(t) \right] dt > 0$$

The statement of (25) is the same contradiction which we have required.
As a result, we have:

$$(26) \quad \frac{d^2V}{d\varepsilon^2}_{(\varepsilon=0)} \leq 0 \ \text{for maximum} \ \rightarrow \quad F_{y'y'} \leq 0$$

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