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The Modified Tanh Method for Solving the Improved Eckhausequation and the (2+1)-dimensional Improved Eckhaus Equation

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Abstract: The modified tanh method is one of most direct and effective algebraic method for obtaining exact solutions of nonlinear partial differential equations. The method can be applied to nonintegrable equations as well as to integrable ones. In this paper,we look for exact soliton solutions of the improved Eckhaus equation and the (2+1)-dimensional improved Eckhaus equation.
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Key words: Modified tanh method; Improved Eckhaus equation; (2+ 1)-dimensional improved Eckhaus equation.

INTRODUCTION

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as, fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-sech method (Malfliet W., 1992; Khater A.H., 2002; Wazwaz AM, 2006), extended tanh method (El-Wakil S.A., 2007; Fan E., 2000; Wazwaz A.M., 2005), hyperbolic function method (Xia T.C., 2001), sine-cosine method (Wazwaz A.M., 2004; Yusufoglu E., 2006) Jacobi elliptic function expansion method (Inc M., 2005), F-expansion method (Sheng Zhang, 2006), and the first integral method (Feng Z.S., 2002; Ding TR, 1997). In recent years,there was interest in obtaining exact solutions of NLPDEs by the extended tanh method. The standard tanh method is developed by Malfliet (1992). Recently, Wazwaz investigated exact solutions of NLPDEs by the extended tanh method. Fan in presented the generalized tanh method for constructing the exact solutions of NLPDEs, such as, the (2 + 1)-dimensional sine-Gordon equation and the double sine-Gordon equation. The Eckhaus equation (M.J. Ablowitz, eads

iut +uxx +2(|u|2)xu+|u|4u=0, u:R! C.

This equation is of nonlinear Schrodinger type. Eckhaus was found in (Calogero, F., 1987) as an asymptotic multi-scale reduction of certain classes of nonlinear partial differential equations. In (Calogero, 1987), many of the properties of the Eckhaus equation were investigated. In (Calogero, 1987), the Eckaus equations was(exactly)linearised by a change(dependent)of variable. The improved Eckhaus equation, and the (2+ 1)-dimensional improved Eckhaus equation are given by

iut+uxx +2(|u|2)xxu+|u|4u=0,

and
iut + uxx - uyy + $2(|u|2)_{xx}$u + |u|4u = 0.

The aim of this paper is to find exact solutions of the improved Eckhaus equation, and the (2+ 1)-dimensional improved Eckhaus equation by the modified extended tanh method.

2. Modified Extended Tanh Method:

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For given a nonlinear equation

$F(u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, ...) = 0,$ (1)

when we look for its traveling wave solutions, the first step is to introduce the wave transformation u *(x,y,t)* = U(ξ), $\xi x + \gamma y + \lambda t$ and change Eq.(1) to an ordinary differential equation(ODE)

$$H(U, U', U'', U''', ...) = 0. \quad (2)$$

The next crucial step is to introduce a new variable ? = ?(?), which is a solution of the Riccati equation

$$\frac{d\phi}{d\varepsilon} = k + \phi^2 \quad (3)$$

The modified extended tanh method admits the use of the finite expansion:

$$u(x, y, t) = U(\xi) = \sum_{i=0}^{m} a_i \phi^i(\xi) + \sum_{i=1}^{m} b_i \phi^{-i}(\xi) \quad (4)$$

where the positive integer m is usually obtained by balancing the highest-order linear term with the nonlinear terms in Eq.(2). Expansion (4) reduces to the generalized tanh method [16] for bi = 0, i = 1, ..., m. Substituting Eq.(3) and Eq.(4) into Eq.(2) and then setting zero all coefficients of ?i(?), we can obtain a system of algebraic equations with respect to the constants k, γ , λ a_0, ..., a_m, b_1, ..., b_m. Then we can determine the constants γ , λ a_0, ..., a_m, b_1, ..., b_m. The Riccati equation (3) has the general solutions:

If k < 0 then

$$\phi(\xi) = -\sqrt{-k}\tanh(\sqrt{-k\xi}),$$
$$\phi(\xi) = -\sqrt{-k}\coth(\sqrt{-k\xi}). \quad (5)$$

If k = 0 then

$$\phi(\xi) = -\frac{1}{\xi} \quad (6)$$

If k > 0 then

$$\phi(\xi) = -\sqrt{k}\tan(\sqrt{-k\xi}),$$
$$\phi(\xi) = -\sqrt{k}\cot(\sqrt{-k\xi}). \quad (7)$$

Therefore, by the sign test of k, we get exact soliton solutions of Eq. (1).

3 Improved Eckhaus Equation:

Let us consider the improved Eckhaus equation

$$iu_t + u_{xx} + 2(|u|^2)_{xx} u + |u|^4 u = 0 \quad (8)$$

where u = u (x,t).

We introduce the wave transformation

$$u(x,t) = e^{i\theta} U(\varepsilon), \theta = \alpha x + \beta t, \xi = x + \lambda t \quad (9)$$

where α, β and λ are real constants and U(?) is real function. Substituting (9) into Eq.(8) we obtain the relation

$$-(\beta+\alpha^2)U+U''+2(U^2)''U+U^5=0$$

hence

$$-(\beta+\alpha^2)U+U''+4(U')^2U+4U''U^2+U^5=0. \tag{10}$$

By balancing the term U00U2 with the term U5 in Eq.(10),we obtain m+2+2m=5m,

$$m+2+2m=5m,$$

then m = 1.

The modified extended tanh method (4) admits the use of the finite expansion:

$$U(\xi)=a_0+a_1\phi(\xi)+\frac{b_1}{\phi(\xi)}. \tag{11}$$

Substituting (11) into Eq.(10), making use of Eq.(3), and by using Maple, we get a system of algebraic equations, for a_0, a_1, b_1, α, β and k in the form

$$\phi^5: 12a_1^3+a_1^5=0,$$
$$\phi^4: 20a_0a_1^2+5a_0a_1^4=0,$$
$$\phi^3: 5b_1a_1^4+16ka_1^3+10a_0^2a_1^3+12a_1^2b_1+8a_0^2a_1+2a_1=0,$$
$$\phi^2: 20b_1a_0a_1^3+24ka_0a_1^2+10a_0^3a_1^2+8b_1a_0a_1=0,$$
$$\phi^1: 4k^2a_1^3+10b_1^2a_1^3+16kb_1a_1^2+30b_1a_0^2a_1^2+4b_1^2a_1$$
$$-(\beta+\alpha^2)a_1+5a_0^4a_1+8ka_0^2a_1+2ka_1=0,$$
$$\phi^0: 30b_1^2a_0a_1^2+4k^2a_0a_1^2+20b_1a_0^3a_1+16ka_0b_1a_1$$
$$-(\beta+\alpha^2)a_0+a_0^5+4b_1^2a_0=0,$$
$$\phi^{-1}: 10b_1^3a_1^2+4k^2b_1a_1^2+16kb_1^2a_1+30b_1^2a_0^2a_1$$
$$+5b_1a_0^4+8kb_1a_0^2+2kb_1+4b_1^3-(\beta+\alpha^2)b_1=0,$$
$$\phi^{-2}: 8k^2b_1a_0a_1+20b_1^3a_0a_1+24kb_1^2a_0+10b_1^2a_0^3=0,$$
$$\phi^{-3}: 12k^2b_1^2a_1+5b_1^4a_1+8k^2b_1a_0^2+16kb_1^3+2k^2b_1+10b_1^3a_0^2=0,$$

4 (2+1)-Dimensional Improved Eckhaus Equation:

$$\phi^{-4}: 5b_1^4a_0+20k^2b_1^2a_0=0,$$
$$\phi^{-5}: 12k^2b_1^3+b_1^5=0.$$

These equations lead to the following cases:

$$k=\frac{1}{96},\quad \beta=\frac{1}{64}-\alpha^2,\quad a_0=a_1=0,\quad b_1=\pm i\frac{\sqrt{3}}{48}, \tag{12}$$

$$k=\frac{1}{96},\quad \beta=\frac{1}{64}-\alpha^2,\quad a_0=0,\quad a_1=\pm 2i\sqrt{3},\quad b_1=0, \tag{13}$$

$$k=\frac{1}{384},\quad \beta=\frac{1}{64}-\alpha^2,\quad a_0=0,\quad a_1=\pm 2i\sqrt{3},\quad b_1=\pm i\frac{\sqrt{3}}{192}, \tag{14}$$

$$k=-\frac{1}{192},\quad \beta=\frac{1}{48}-\alpha^2,\quad a_0=0,\quad a_1=\pm 2i\sqrt{3},\quad b_1=\pm i\frac{\sqrt{3}}{96}, \tag{15}$$

where α is an arbitrary constant.
The sets (12) - (15) give the soliton solutions

$$u_1(x,t)=\pm\frac{i}{2\sqrt{2}}e^{i(\alpha x+(\frac{1-64\alpha^2}{64})t)}cot(\frac{1}{4\sqrt{6}}(x-2\alpha t)).$$

$$u_2(x,t)=\pm\frac{i}{2\sqrt{2}}e^{i(\alpha x+(\frac{1-64\alpha^2}{64})t)}tan(\frac{1}{4\sqrt{6}}(x-2\alpha t)).$$

$$\begin{aligned} u_3(x,t) &= \pm\frac{i}{4\sqrt{2}}e^{i(\alpha x+(\frac{1-64\alpha^2}{64})t)}\{tan(\frac{1}{8\sqrt{6}}(x-2\alpha t) \\ &+ cot(\frac{1}{8\sqrt{6}}(x-2\alpha t)\}. \end{aligned}$$

$$\begin{aligned} u_4(x,t) &= \pm\frac{i}{4}e^{i(\alpha x+(\frac{1-48\alpha^2}{48})t)}\{tanh(\frac{1}{8\sqrt{3}}(x-2\alpha t) \\ &+ coth(\frac{1}{8\sqrt{3}}(x-2\alpha t)\}. \end{aligned}$$

4 (2+1)-Dimensional Improved Eckhaus Equation:

In this section we study the (2+1)-dimensional improved Eckhaus equation

$$iu_t+u_{xx}-u_{yy}+2(|u|^2)_{xx}u+|u|^4u=0. \tag{16}$$

where u = u (x,y,t).
Let us consider the traveling wave solutions

$$u(x,y,t)=e^{i\theta}U(\xi),\quad \theta=\alpha x+\beta y+\delta t,\quad \xi=x+\gamma y+\lambda t, \tag{17}$$

where *α, β, δ, γ, λ* are real constants and U(ζ) is real function. Substituting (17) into Eq.(16), we find the relation λ = -2(*α* - *β γ*) , then (16) is following nonlinear ordinary differential equation

$$(\beta^2-\alpha^2-\delta)U+(1-\gamma^2)U''+2(U^2)''U+U^5=0. \tag{18}$$

4 (2+1)-Dimensional Improved Eckhaus Equation:

It is easy to show that m = 1, if balancing U00U2 with U5.
The modified extended tanh method (4) admits the use of the finite expansion:

$$U(\xi) = a_0 + a_1\phi(\xi) + \frac{b_1}{\phi(\xi)}. \tag{19}$$

Substituting Eqs.(3) and (19) into Eq.(18), and equating the coefficients of like powers of ?i(i = -5, -4, -3, -2, -1,0, 1,2,3,4,5) to zero yields the system of algebraic equations to a_0, a_1, b_1, α, δ, β, γ and k

$$\phi^5 : 12a_1^3 + a_1^5 = 0,$$
$$\phi^4 : 20a_0a_1^2 + 5a_0a_1^4 = 0,$$
$$\phi^3 : 5b_1a_1^4 + 16ka_1^3 + 10a_0^2a_1^3 + 12a_1^2b_1 + 8a_0^2a_1 + 2(1-\gamma^2)a_1 = 0,$$
$$\phi^2 : 20b_1a_0a_1^3 + 24ka_0a_1^2 + 10a_0^3a_1^2 + 8b_1a_0a_1 = 0,$$
$$\phi^1 : 4k^2a_1^3 + 10b_1^2a_1^3 + 16kb_1a_1^2 + 30b_1a_0^2a_1^2 + 4b_1^2a_1$$
$$+(\beta^2-\alpha^2-\delta)a_1 + 5a_0^4a_1 + 8ka_0^2a_1 + 2(1-\gamma^2)ka_1 = 0,$$
$$\phi^0 : 30b_1^2a_0a_1^2 + 4k^2a_0a_1^2 + 20b_1a_0^3a_1 + 16ka_0b_1a_1$$
$$+(\beta^2-\alpha^2-\delta)a_0 + a_0^5 + 4b_1^2a_0 = 0,$$
$$\phi^{-1} : 10b_1^3a_1^2 + 4k^2b_1a_1^2 + 16kb_1^2a_1 + 30b_1^2a_0^2a_1$$
$$+5b_1a_0^4 + 8kb_1a_0^2 + 2(1-\gamma^2)kb_1 + 4b_1^3 + (\beta^2-\alpha^2-\delta)b_1 = 0,$$
$$\phi^{-2} : 8k^2b_1a_0a_1 + 20b_1^3a_0a_1 + 24kb_1^2a_0 + 10b_1^2a_0^3 = 0,$$
$$\phi^{-3} : 12k^2b_1^2a_1 + 5b_1^4a_1 + 8k^2b_1a_0^2 + 16kb_1^3 + 2(1-\gamma^2)k^2b_1 + 10b_1^3a_0^2 = 0,$$
$$\phi^{-4} : 5b_1^4a_0 + 20k^2b_1^2a_0 = 0,$$
$$\phi^{-5} : 12k^2b_1^3 + b_1^5 = 0.$$

These equations lead to the following cases:
Case A:

$$k = \frac{1}{96}(1-\gamma^2), \quad \delta = \beta^2 - \alpha^2 + \frac{1}{64} - \frac{1}{32}\gamma^2 + \frac{1}{64}\gamma^4, \quad a_0 = 0, \quad a_1 = \pm 2i\sqrt{3}, \quad b_1 = 0,$$

with *α, β, γ* being arbitrary constants. By using (6) and (7) and (5) we obtain three solutions, namely,

$$u_1(x,y,t) = \pm\frac{2\sqrt{3}i}{x \pm y - 2(\alpha \pm \beta)t}e^{i(\alpha x + \beta y + (\beta^2-\alpha^2)t)},$$

for k = 0.

$$u_2(x,y,t) = \pm\frac{i\sqrt{1-\gamma^2}}{2\sqrt{2}}e^{i(\alpha x+\beta y+(\beta^2-\alpha^2+\frac{1}{64}-\frac{1}{32}\gamma^2+\frac{1}{64}\gamma^4)t)}tan(\frac{\sqrt{1-\gamma^2}}{4\sqrt{6}}(x+\gamma y - 2(\alpha-\beta\gamma)t)),$$

for k> 0.

$$u_3(x,y,t) = \pm\frac{i\sqrt{\gamma^2-1}}{2\sqrt{2}}e^{i(\alpha x+\beta y+(\beta^2-\alpha^2+\frac{1}{64}-\frac{1}{32}\gamma^2+\frac{1}{64}\gamma^4)t)}tanh(\frac{\sqrt{\gamma^2-1}}{4\sqrt{6}}(x+\gamma y - 2(\alpha-\beta\gamma)t)),$$

for k < 0. Case B:

$$k = \frac{1}{96}(1-\gamma^2), \quad \delta = \beta^2 - \alpha^2 + \frac{1}{64} - \frac{1}{32}\gamma^2 + \frac{1}{64}\gamma^4, \quad a_0 = a_1 = 0, \quad b_1 = \pm i\frac{\sqrt{3}}{48}(1-\gamma^2),$$

with α, β, γ being arbitrary constants. By using (7) and (5) we obtain two solutions, namely,

$$u_4(x,y,t) = \pm\frac{i\sqrt{1-\gamma^2}}{2\sqrt{2}}e^{i(\alpha x+\beta y+(\beta^2-\alpha^2+\frac{1}{64}-\frac{1}{32}\gamma^2+\frac{1}{64}\gamma^4)t)}cot(\frac{\sqrt{1-\gamma^2}}{4\sqrt{6}}(x+\gamma y-2(\alpha-\beta\gamma)t)),$$

for $k > 0$.

$$u_5(x,y,t) = \pm\frac{i\sqrt{\gamma^2-1}}{2\sqrt{2}}e^{i(\alpha x+\beta y+(\beta^2-\alpha^2+\frac{1}{64}-\frac{1}{32}\gamma^2+\frac{1}{64}\gamma^4)t)}coth(\frac{\sqrt{\gamma^2-1}}{4\sqrt{6}}(x+\gamma y-2(\alpha-\beta\gamma)t)),$$

for $k < 0$.

5. Conclusion:

In this paper, the modified tanh method has been successfully applied to find the solution for two nonlinear partial differential equations such as improved Eckhaus and (2 + 1)-dimensional improved Eckhaus equations. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

REFERENCES

Ablowitz, M.J., G. Biondini, S. De Lillo, 1997. On the well-posedness of the Eckhaus equa-tion. Physics Letters A 230: 319-323.

Calogero, F., W. Eckhaus, 1987. Nonlinear evolution equations, rescalings,model PDES and their integrability: I. Inv Probl., 3: 229-262.

Calogero, F., S. De Lillo, 1987. The Eckhause PDE $iu_t + u_{xx} + 2(|u|^2)_{xx}u + |u|^4.u = 0$. Inv Probl 4: 633-682.

Calogero, F., 1987. The evolution partial differential equation ut = uxxx + 3(uxxu2 + 3u2 xu) + 3uxu4. J Math Phys., 28: 538-555.

Ding T.R., C.Z. Li, 1996. Ordinary differential equations.Peking University Press,Peking,

El-Wakil S.A., M.A. Abdou, 2007. New exact travelling wave solutions using modified ex-tended tanh-function method. Chaos Solitons Fractals., 31(4): 840-852.

Fan E., 2000. Extended tanh-function method and its applications to nonlinear equations. Phys Lett A, 277(4-5): 212-218.

Feng Z.S., 2002. The first integer method to study the Burgers-Korteweg-de Vries equation, J Phys A: Math. Gen., 35(2): 343-349.

Fan, E., Y.C. Hon, 2002. Generalized tanh method extended to special types of nonlinear equations, Z. Naturforsch, 57a: 692-700.

Inc M., M. Ergut, 2005. Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method. Appl Math E-Notes, 5: 89-96.

Khater A.H., W. Malfliet, D.K. Callebaut and E.S. Kamel, 2002. The tanh method,a simple transformation and exact analytical solutions for nonlinear reactiondiffusion equa-tions. Chaos Solitons Fractals, 14(3): 513-522.

Malfliet W., 1992. Solitary wave solutions of nonlinear wave equations. Am J Phys., 60(7): 650-654.

Sheng Zhang., 2006. The periodic wave solutions for the (2 + 1)-dimensional Konopelchenko Dubrovsky equations. Chaos Solitons Fractals., 30: 1213-1220.

Wazwaz A.M., 2006. Two reliable methods for solving variants of the KdV equation with compact and noncompact structures. Chaos Solitons Fractals., 28(2): 454-462.

Wazwaz A.M., 2005. The tanh-function method: Solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations. Chaos Soli-tons and Fractals., 25(1): 55-63.

Wazwaz A.M., 2004. The sine-cosine method for obtaining solutions with compact and non-compact structures. Appl Math Comput., 159(2): 559-576.

Wazwaz A.M., 2004. A sine-cosine method for handling nonlinear wave equations. Math Comput Modelling., 40(5-6): 499-508.

Xia T.C., B. Li and H.Q. Zhang, 2001. New explicit and exact solutions for the Nizhnik-Novikov-Vesselov equation. Appl Math E-Notes., 139-142.

Yusufoglu E., A. Bekir, 2006. Solitons and periodic solutions of coupled nonlinear evolution equations by using Sine-Cosine method. Internat J Comput Math., 83(12): 915-924.