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Numerical Solution of Linear Fredholm Fuzzy Integral Equations by Modified Homotopy Perturbation Method

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Abstarct: By using parametric form of fuzzy numbers we convert Fredholm Fuzzy Integral Equation to a linear system of integral equations of the second kind in crisp case.We use Modified homotopy perturbation method and find the approximate solution of this system and hence obtain an approximation for fuzzy solution of the linear fuzzy Fredholm integral equation.

Key words: Fuzzy integral equations, Homotopy perturbation method

INTRODUCTION

The topics of fuzzy integral equations (FIE) which growing interest for sometime, in particular in relation to fuzzy control, have been rapidly developed in recent years. Prior to discussing fuzzy differential and integral equations and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus.The concept of fuzzy sets, was originally introduced by Zadeh (1975), led to the definition of fuzzy numbers and its implementation in fuzzy control and approximate reasoning problems. The basic arithmetic structure for fuzzy numbers was later developed by Dubois and Prade (1978), all of them observed fuzzy numbers as a collection of alevels. Additional related material can be found in. Goetschel and Voxman (1986) suggested a new approach. They represented fuzzy number as a parameterized triple (see Section 2) and then embedded the set of fuzzy numbers into a topological vector space. This enabled them to design the basics of a fuzzy calculus.

2 Preliminaries:

In this section the most basic notations used in fuzzy calculus are introduced.

Definition 2.1:

A fuzzy number is a fuzzy set $\tilde{u}: R \to I = [0,1]$ which satisfies

1. $\tilde{u}$ is upper semi continuous.

2. $\tilde{u}(x) = 0$ outside some interval $[c,d]$

3. there are real number $a,b : c \le a \le b \le d$ for which:

1.1 $\tilde{u}(x)$ is monotonic increasing on $[c,a]$

1.2 $\tilde{u}(x)$ is monotonic decreasing on $[b,d]$

1.3 $\tilde{u}(x) = 1, a \le x \le b$

An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva. A fuzzy number $\tilde{u}$ is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r), \overline{u}(r); 0 \le r \le 1$ which satisfy the following

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requirements:

1. $\underline{u}(r)$ is monotonically increasing left continuous function .

2. $\overline{u}(r)$ is monotonically decreasing left continuous function.

3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A popular fuzzy number is trapezoidal fuzzy number with tolerance interval $[a,b]$, left width α and right width β we use the notation:

$\tilde{u} = (a, b, \alpha, \beta)$

Its parametric form:

$$\underline{u}(r) = a - (1-r)\alpha \quad , \quad \overline{u}(r) = b + (1-r)\beta$$

If $a = b$ then trapezoidal transform to triangular fuzzy number and we denote all of the triangular fuzzy number with $\mathbf{FT}(R)$. Let $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, $\tilde{u} = (\underline{u}(r), \overline{u}(r))$. Some results of applying fuzzy arithmetics on fuzzy numbers $\tilde{v}$, $\tilde{u}$ are as follows: (Zadeh, L.A., 1975)

- ;$x > 0 : x = (x\underline{v}(r), x\overline{v}(r))$
- $x < 0 : x = (x\overline{v}(r), x\underline{v}(r))$
- $\tilde{v} + \tilde{u} = (\underline{v}(r) + \underline{u}(r), \overline{v}(r) + \overline{u}(r))$
- $\tilde{v} - \tilde{u} = (\underline{v}(r) - \overline{u}(r), \overline{v}(r) - \underline{u}(r))$

Definition 2.2(Gal, S.G., 2000)
For arbitrary fuzzy numbers $(\underline{u}(r), \overline{u}(r))$, $(\underline{v}(r), \overline{v}(r))$ the quantity

$$D(\tilde{u}, \tilde{v}) = max\left\{ \sup_{0 \leq r \leq 1} |\, \underline{v}(r) - \underline{u}(r) \,|, \sup_{0 \leq r \leq 1} |\, \overline{v}(r) - \overline{u}(r) \,| \right\}$$

is the distance between $\tilde{u}$ and $\tilde{v}$.
If the fuzzy function *f(t)* is continuous in the metric *D*, its definite integral exists. Furthermore

$$\underline{\left(\int_a^b f(t,r)dt \right)} = \int_a^b \underline{f}(t,r)dt$$
$$\overline{\left(\int_a^b f(t,r)dt \right)} = \int_a^b \overline{f}(t,r)dt \tag{1}$$

3 Fuzzy Integral Equation:
The integral equations which are discussed in this section are the Fredholm equations of the second kind(FFIE-2) as follow:

$$u(t) = f(t) + \beta \int_a^b K(s,t)u(s)ds \tag{2}$$

where $\beta > 0, K(s,t)$ is an arbitrary kernel function over the square $a \le s,t \le b$ and *f(t)* is a function of $t : a \le t \le b$. If *f(t)* is a crisp function then the solutions of Eq. (2) are crisp as well. However, if *f(t)* is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind,
i.e. to Eq. (2) where *f(t)* is a fuzzy function, are given in(Guerra, M.L., L. Stefanini, 2005).
Now we introduce parametric form of a FFIE-2 with respect to Definition 2.1.

Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{u}(t,r), \overline{u}(t,r))$, $0 \le r \le 1$ and $t \in [a,b]$ are parametric form of *f(t)* and *u(t)*, respectively then, parametric form of FFIE-2 is as follows:

$$\begin{aligned} \underline{u}(t,r) &= \underline{f}(t,r) + \beta \int_a^b v_1(s,t,\underline{u}(s,r),\overline{u}(s,r))ds \\ \overline{u}(t,r) &= \overline{f}(t,r) + \beta \int_a^b v_2(s,t,\underline{u}(s,r),\overline{u}(s,r))ds \end{aligned} \tag{3}$$

where

$$v_1(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} K(s,t)\underline{u}(s,r) & K(s,t) \ge 0 \\ K(s,t)\overline{u}(s,r) & K(s,t) < 0 \end{cases}$$

and

$$v_2(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} K(s,t)\overline{u}(s,r) & K(s,t) \ge 0 \\ K(s,t)\underline{u}(s,r) & K(s,t) < 0 \end{cases}$$

for each $0 \le r \le 1$ and $a \le t \le b$. In next section, we explain Homotopy perturbation method(HPM) as a numerical algorithm for approximating of linear integral equations in crisp case then, we find approximate solutions for $\underline{u}(t,r)$ and $\overline{u}(t,r)$ for each $0 \le r \le 1$ and $a \le t \le b$.

4 Modify HPM:

To illustrate the HPM, Ji-Huan He (1999) considered the following nonlinear differential equation:

$$A(u) = f(r), \quad r \in \Omega \tag{4}$$

with boundary conditions

$$\mathbf{B} = (u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \tag{5}$$

Where *A* is a general differential operator, *B* is a boundary operator, *f(r)* is a known analytic function, Γ is the boundary of the domain. Suppose the operator *A* can be divided into two parts: *M* and *N*. Therefore, (3) can be rewritten as follows:

$$M(u) + N(u) = f(r) \tag{6}$$

He in (1999) constructed a homotopy $v(r,p) : \Gamma \times [0,1] \to R$ which satisfies

$$H(\nu,p)=M(\nu)-M(y_0)+pM(y_0)+p[N(\nu)-f(r)]=0 \tag{7}$$

where $r\in\Omega$ and $p\in(0,1]$ is an imbedding parameter, and y_0 is an initial approximation of (3). Hence, it is easy to see

$$H(\nu,0)=M(\nu)-M(y_0)=0,$$
$$H(\nu,p)=A(\nu)-f(r)=0$$

and changing the variation of p from 0 to 1 is the same as changing $H(\nu,p)$ from $M(\nu)-M(y_0)$ to $A(\nu)-f(r)$. In topology, this is called deformation, $M(\nu)-M(y_0)$ and $A(\nu)-f(r)$ are called homotopic. Owing to the fact that $0\le p\le 1$ can be considered as a small parameter, by applying the perturbation technique, we can assume that the solution of (6) and (7) can be expressed as a series in *p*, as follows:

$$\nu=\nu_0+p\nu_1+p^2\nu_2+\ldots \tag{8}$$

when $p\to 1$, Eqs. (6) and (7) correspond to Eqs. (5) and (8) becomes the approximate solution of Eq. (5), i.e.

$$u(x)=\lim_{p\to 1}\nu=\nu_0+\nu_1+\nu_2+\ldots \tag{9}$$

The series (9) is convergent for most of the cases, and also the rate of convergence depends on how we choose *A(v)*.

Now we propose a scheme to accelerate the rate of convergence of HPM applied to linear Fredholm integral equations with kernels of the form $K(x,t)=g(x)h(t)$. We define a new convex homotopy perturbation as follows:[5]

$$H(u,p,m)=(1-p)F(u)+pL(u)+p(1-p)\left[mg(x)\right] \tag{10}$$

where

$$F(u)=u(x)-f(x)$$

and

$$L(u)=u(x)-f(x)-\int_a^b g(x)h(t)u(t)dt=0$$

hence we can write

$$u-f-pg(x)\int_a^b h(t)u(t)dt+mpg(x)-mp^2g(x)=0 \tag{11}$$

now by using (3) in (7), and equating the terms with identical power of *p*, we obtain

$$p^0 : u_0 - f(x) = 0 \Rightarrow u_0 = f(x)$$

$$p^1 : u_1 + mg(x) - g(x)\int_a^b h(t)u_0(t)dt = 0,$$

$$u_1 = (c-m)g(x), \quad c = \int_a^b h(t)f(t)dt$$

$$p^2 : u_2 - mg(x) - g(x)\int_a^b h(t)u_1(t)dt = 0$$

$$u_2 = [m+(c-m)\alpha]g(x), \quad \alpha = \int_a^b h(t)g(t)dt$$

$$p^3 : u_3 = \int_a^b h(t)u_2(t)dt$$

and in general

$$u_{n+1} = \int_a^b h(t)u_n(t)dt, \quad n = 2,3,...$$

now we find m such that $u_2 = 0$, since if $u_2 = 0$ then $u_3 = u_4 = ... = 0$, and the exact solution will be obtained as $u(x) = u_0(x) + u_1(x)$, hence for all values of x we should have

$$m + (c-m)\alpha = 0$$

or

$$m = \frac{c\alpha}{\alpha - 1} = \frac{[\int_a^b h(t)f(t)dt][\int_a^b h(t)g(t)dt]}{\int_a^b h(t)g(t)dt - 1}$$

or

$$m = \frac{\int_a^b k(t,t)dt}{\int_a^b k(t,t)dt - 1}\int_a^b h(t)f(t)dt$$

provided that

$$\int_a^b k(t,t)dt \neq 1.$$

Now consider the general case

$$k(x,t) = \sum_1^N g_i(x)h_i(t)$$

here we choose the convex homotopy as follows:

$$H(u,p,m) = (1-p)F(u) + pL(u) + p(1-p)\left[\sum_1^N m_i g_i(x)\right] = 0$$

by doing similar manipulations, we obtain

$$
\begin{aligned}
u_0 &= f \\
u_1 &= -\sum_1^N m_i g_i(x) + \sum_1^N g_i(x) \int_a^b h_i(t) u_0(t) dt \\
&= \sum_1^N \left[\int_a^b h_i f(t) dt - m_i \right] g_i(x) * 3mm \\
u_2 &= \sum_1^N m_i g_i(x) + \sum_1^N g_i(x) \int_a^b h_i(t) u_1(t) dt \\
&= \sum_1^N \left[m_i + \int_a^b h_i u_1 dt \right] g_i(x) \\
&\vdots \\
u_{n+1} &= \sum_1^N g_i(x) \int_a^b h_i(t) u_n(t) dt, \quad i = 2,3,...,
\end{aligned}
$$

we try to find the parameters m_i , $i = 1,2,...,N$, such that

$$u_2 = u_3 = ... = 0$$

hence from u_2 we should have

$$m_i + \int_a^b h_i(t) u_1(t) dt = 0$$

now by substituting u_1 in (9), we obtain

$$m_i + \sum_{j=1}^N \int_a^b h_i(t) \left[\int_a^b h_j(t) f(t) dt - m_j \right] g_j(t) dt = 0$$

let

$$c_j = \int_a^b h_j(t) f(t) dt, \quad b_{ij} = \int_a^b h_j(t) g_j(t) dt$$

then

$$m_i + \sum_{j=1}^N b_{ij} (c_j - m_j) = 0, \quad i = 1,2,...,N \tag{12}$$

under certain condition,the values of $m_i, i = 1,2,...,N$, can be obtained from the system of linear equations in (10). Let the matrix $\mathbf{B}$ and the vectors $\mathbf{m}$ and $\mathbf{c}$ be defined as follows:

$$\mathbf{B} = [b_{ij}], \quad \mathbf{m} = [m_j], \quad \mathbf{c} = [c_j]$$

therefore from (10) we can write

$$(\mathbf{B} - \mathbf{I})\mathbf{m} = \mathbf{Bc}$$

and if $(\mathbf{B} - \mathbf{I})$ is nonsingular then

$$\mathbf{m} = (\mathbf{B} - \mathbf{I})^{-1}\mathbf{Bc}$$

In the case of non-degenerate kernels,by using Taylor expansion for functions of two variables, we can write $k(x,t)$ (if possible) as follows:

$$k(x,t) = \sum_{i=1}^{N} g_i(x) h_i(t)$$

and by applying $H(u,p,m)$ method we can approximate the solution of the given integral equation.

5 Numerical Examples:
Example 5.1:

Consider the fuzzy Fredholm integral equation with

$$\underline{f}(t,r) = t^2(\frac{3}{15}(r^2+r) + \frac{2}{15}(4-r^3-r))$$
$$\overline{f}(t,r) = t^2(\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r))$$

and kernel

$$K(s,t) = st \qquad 0 \le s,t \le 1$$

The exact solution in this case is given by

$$\underline{u}(t,r) = (t^2 + \frac{3}{8}t)(\frac{3}{15}(r^2+r) + \frac{2}{15}(4-r^3-r))$$

$$\overline{u}(t,r) = (t^2 + \frac{3}{8}t)(\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r))$$

Some first terms of MHPM series are:

$$\underline{u}_0(t,r) = \frac{1}{15}t^2(3(r^2+r) + 2(4-r^3-r))$$

$$\underline{u}_1(t,r) = \frac{3}{120}t(3(r^2+r) + 2(4-r^3-r))$$

$$\underline{u}_2(t,r) = 0$$

and

$$\overline{u}_0(t,r) = \frac{1}{15}t^2\,\frac{2}{15}(r^2+r) + \frac{13}{15}(4-r^3-r))$$

$$\bar{u}_1(t,r) = \frac{3}{120} t(\frac{2}{15}(r^2 + r) + \frac{13}{15}(4 - r^3 - r))$$

$$\bar{u}_2(t,r) = 0$$

As we observe, after two terms the exact solution is obtained.

Conclusion:

In this work we illustrated a numerical algorithm for solving fuzzy Fredholm integral equations of the second kind, using Modify HPM method. We feel that this work which presents applicable computational methods, may help to narrow the existing gap between the theoretical research on FIEs.

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REFERENCES

Abbasbandy, S., 2007. Application of He’s homotopy pertrubation method to functional integral equations. Chaos,solitons & Fractals, 31(5): 1243-7.

He, J.H., 1999. Homotopy pertrubation technique. Comput Method Appl Mech Eng., 173(3-4): 257-62.

Dubois, D., H. Prade, 1982. Towards fuzzy differential calculus: Part 3, differentiation, Fuzzy Sets and Systems, 8: 225-233.

Gal, S.G., 2000. Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman Hall CRC Press, 617-666.

Golbabai, A., B. Keramati, 2008. Modified homotopy perturbation method for solving Fredholm integral equations, Chaos, Soliton and Fractals, 37: 1528-1537.

Guerra, M.L., L. Stefanini, 2005. "Approximate fuzzy arithmetic operation using monotonic interpolation", Fuzzy Sets and Systems, 150: 5-33.

Friedman, M., M. Ming, A. Kandel, 1998. Fuzzy linear systems, Fuzzy Sets and Systems, 96: 201-209.

Zadeh, L.A., 1975. "The concept of a linguistic variable and its application to approximate reasoninig", Information Sciences, 8: 199-249.

Badard, R., 1984. Fixed point theorem for fuzzy numbers Fuzzy sets systems, 13: 291-302.

Dubois, D., H. Prade, 1978. Operation on fuzzy numbers, J. System Sci., 9: 613-626.

Goetschel, R., W. Voxman, 1986. Elemtary Calculus, Fuzzy sets system, 18: 31-43.

Wu, C., M. Ma, 1990. On the integrals, series and integral equations of fuzzy set-valued functions, J. Harbin Inst. Technol., 21: 11-19.