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Fuzzy Approximately Additive - Cubic Functional Equations

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Abstract: In this paper we I nvestigate the generalized Hyers–Ulam stability of the additive–cubic functional equation $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x)$ in fuzzy normed spaces.

Key word:

Introduction and Preliminaries:

In order to construct a fuzzy structure on a linear space, A.K. Katsaras (Khodaei, 2010) defined the notion of fuzzy norm on a linear space. Since then, a few mathematicians have introduced and discussed several notions of fuzzy norm from different points of view (Krishna, 1994; Xiao, 2003). In particular, T. Bag and S.K. Samanta (2003), gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type (Kramosil, 1975). They also studied some nice properties of the fuzzy norm in (Bag, 2005).

Defining, in some way, the class of approximate solutions of the given functional equation, one can ask whether each mapping from this class can be somehow approximated by an exact solution of the considered equation (Gordji, 2003; Hyers, 1998; Khodaei, 2010; Park, 2010; Park, 2010; Shakeri, 2010). In 1940 Ulam (1960) posed the first stability problem. In the next year, Hyers (2008) gave a partial affirmative answer to the question of Ulam. Hyers theorem was generalized by Aoki (1950) for additive mappings by considering an unbounded Cauchy difference. On the other hand J.M. Rassias (1989, 1985, 1984, 1982) generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J.M. Rassias Theorem:

Theorem 1.1: If it is assumed that there exist constants $\varepsilon \geq 0$ and $p_1, p_2 \in R$ such that $p = p_1 + p_2 \neq 1$, and $f: E \to E'$ is a map from a norm E space into a Banach space E` such that the inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon \|x\|^{p_1} \|y\|^{p_2}$$

for all $x, y \in E$, then there exists a unique additive mapping $T: E \to E'$ such that

$$\|f(x)-T(x)\| \leq \frac{\varepsilon}{2-2^p} \|x\|^p,$$

for all $x \in E$. If in addition for every $x \in E$, $f(tx)$ is continuous in $t \in R$ for each fixed x, then T is linear.

K. Jun and H. Kim (2002) introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

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and they established the general solution and the generalized Hyers–Ulam stability problem for the functional equation (1.1). They proved that a function $f: E_1 \to E_2$ satisfies the functional equation (1.1) if and only if there exists a function $B: E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x,x,x)$ for all $x \in E_1$, and B is symmetric for each fixed one variable and additive for each fixed two variables. The function is given by

$$B(x,y,z) = \frac{1}{24}[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)]$$

for all $x,y,z \in E_1$.

It is easy to see that the function $f(x) = cx^3$ is a solution of the functional equation (1.1). Thus, it is natural that (1.1) is called a cubic functional equation and every solution of the cubic functional equation (1.1) is said to be a cubic function. K. Jun and H. Kim (2006), investigated the generalized Hyers–Ulam stability for a mixed type cubic and additive functional equation.

The authors of this paper (Eshaghi Gordji,), established the generalized Hyers–Ulam stability for a functional equation deriving from additive and quadratic functions in fuzzy Banach spaces. In this paper, we deal with the following functional equation deriving from cubic and additive mappings:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x) \tag{1.2}$$

With $f(0) = 0$.

It is easy to see that the mapping $f(x) = ax^3 + cx$ is a solution of the functional equation (1.2).

The main purpose of this paper is to investigate the generalized Hyers–Ulam stability for Eq. (1.2) in fuzzy normed spaces.

Following (Bag, 2003), we give the following notion of a fuzzy norm.

Let X be a real linear space. A function $N: X \times R \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in R$:

(N_1) $N(x,a) = 0$ for $a \leq 0$;

(N_2) $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;

(N_3) $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;

(N_4) $N(x+y, a+b) \geq \min\{N(x,a), N(y,b)\}$;

(N_5) $N(x,.)$ is non-decreasing function on R and $\lim_{a\to\infty} N(x,a) = 1$;

(N_6) For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on R.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement "the norm of x is less than or equal to the real number a".

Example 1.2. Let $(X,\|.\|)$ be a normed linear space. Then

$$N(x,a)=\begin{cases}\dfrac{a}{a+\|x\|}, & a>0, x\in X,\\ 0, & a\le 0, x\in X\end{cases}$$

is a fuzzy norm on *X*.

Definition 1.3. Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in *X*. Then x_n is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty} N(x_n-x,a)=1$ for all $a>0$. In that case, *x* is called the limit of the sequence x_n and we denote it by $N-\lim_{n\to\infty} x_n = x$.

Definition 1.4. A sequence x_n in *X* is called Cauchy if for each $\varepsilon>0$ and each $a>0$ there exists n_0 such that for all $n\ge n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,a)>1-\varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Main Result:

Throughout this section, assume that *X*, (Y,N) and (Z,N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. For convenience, we use the following abbreviation for a given mapping $f:X\to Y$:

$$Df(x,y):=f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-2f(2x)+4f(x)$$

for all $x,y\in X$.

We start our investigations with fuzzy stability of the functional equation (1.2).

Theorem 2.1. Let $\varphi_1: X\times X\to Z$ be a function such that for some $0<\alpha<2$

$$N'(\varphi_1(2x,2y),a)\ge N'(\alpha\varphi_1(x,y),a) \tag{2.1}$$

for all $x\in X, y\in\{x,2x\}$ and $a>0$, and $\lim_{n\to\infty} N'(\varphi_1(2^n x,2^n y),2^n a)=1$ for all $x,y\in X$ and $a>0$. Suppose that a mapping $f:X\to Y$ with $f(0)=0$ satisfies the inequality

$$N(Df(x,y),a)\ge N'(\varphi_1(x,y),a) \tag{2.2}$$

for all $a>0$ and all $x,y\in X$. Then the limit

$$A(x)=N-\lim_{n\to\infty}\frac{1}{2^n}[f(2^{n+1}x)-8f(2^n x)]$$

exists for all $x\in X$ and the mapping $A:X\to Y$ is a unique additive mapping satisfying

$$N(f(2x)-8f(x)-A(x),a)\geq N''_1(x,\frac{a(2-\alpha)}{3}) \tag{2.3}$$

for all $x\in X$ and all $a>0$, where

$$N''_1(x,a):=\min\{N'(\varphi_1(0,x),a),N'(\varphi_1(x,x),a),N'(\varphi_1(x,2x),a)\}. \tag{2.4}$$

Proof. Letting $x=0$ in (2.2), we get

$$N(f(y)+f(-y),a)\geq N'(\varphi_1(0,y),a) \tag{2.5}$$

for all $y\in X$ and $a>0$. Replacing y by x and $2x$ in (2.2), respectively, we get the inequalities

$$N(f(3x)-4f(2x)+f(x),a)\geq N'(\varphi_1(x,x),a), \tag{2.6}$$

$$N(f(4x)-2f(3x)-2f(2x)-2f(-x)+4f(x),a)\geq N'(\varphi_1(x,2x),a) \tag{2.7}$$

for all $x\in X$ and $a>0$. It follows from (2.5), (2.6) and (2.7),

$$N(f(4x)-10f(2x)+16f(x),3a)\geq N''_1(x,a) \tag{2.8}$$

for all $x\in X$ and $a>0$. Let $g:X\to Y$be a mapping defined by $g(x)=f(2x)-8f(x)$ for all $x\in X$. Therefore, by (2.4) and (2.8) we have

$$N(g(2x)-2g(x),3a)\geq N''_1(x,a) \tag{2.9}$$

for all $x\in X$ and all $a>0$. Thus

$$N(\frac{g(2x)}{2}-g(x),\frac{3a}{2})\geq N''_1(x,a) \tag{2.10}$$

for all $x\in X$ and $a>0$. Replacing x by $2^n x$ in (2.10), we get

$$N(\frac{g(2^{n+1}x)}{2}-g(2^n x),\frac{3a}{2})\geq N''_1(2^n x,a) \tag{2.11}$$

for all $x\in X$ and $a>0$. Using (2.1) we get

$$N(\frac{g(2^{n+1}x)}{2}-g(2^n x),\frac{3a}{2})\geq N''_1(x,\frac{a}{\alpha^n}) \tag{2.12}$$

for all $x\in X$ and $a>0$. Replacing a by $\alpha^n a$ in (2.10), we see that

$$N(\frac{g(2^{n+1}x)}{2^{n+1}}-\frac{g(2^n x)}{2^n},\frac{3a\alpha^n}{2(2^n)})\geq N''_1(x,a) \tag{2.13}$$

for all $x \in X$ and $a > 0$. It follows from $\frac{g(2^n x)}{2^n} - g(x) = \sum_{i=0}^{n-1} \frac{g(2^{i+1}x)}{2^{i+1}} - \frac{g(2^i x)}{2^i}$ and (2.13) that

$$N(\frac{g(2^n x)}{2^n} - g(x), \sum_{i=0}^{n-1} \frac{3a\alpha^i}{2(2^i)}) \geq \min \bigcup_{i=0}^{n-1} \{N(\frac{g(2^{i+1}x)}{2^{i+1}} - \frac{g(2^i x)}{2^i}, \frac{3a\alpha^i}{2(2^i)})\} \geq N''_1(x,a) \tag{2.14}$$

for all $x \in X$ and $a > 0$. Replacing x by $2^m x$ in (2.14), we observe that

$$N(\frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{3a\alpha^i}{2(2^{i+m})}) \geq N''_1(2^m x, a) \geq N''_1(x, \frac{a}{\alpha^m}),$$

Whence

$$N''_1(\frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^m x)}{2^m}, \sum_{i=m}^{n+m-1} \frac{3a\alpha^i}{2(2^i)}) \geq N''_1(x,a)$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$.
Hence

$$N(\frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^m x)}{2^m}, a) \geq N''_1(x, \frac{a}{\sum_{i=m}^{n+m-1} \frac{3\alpha^i}{2(2^i)}}) \tag{2.15}$$

for all $x \in X, a > 0$ and $m, n \geq 0$ Since $0 < \alpha < 2$ and $\sum_{i=0}^{\infty} (\frac{\alpha}{2})^i < \infty$ the Cauchy criterion for convergence and (N_5) imply that $\{\frac{g(2^n x)}{2^n}\}$ is a Cauchy sequence in (Y, N) . Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So one can define the mapping $A: X \to Y$ by $A(x) := N - \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$ for all $x \in X$. Letting $m = 0$ in (2.15), we get

$$N(\frac{g(2^n x)}{2^n} - g(x), a) \geq N''_1(x, \frac{a}{\sum_{i=0}^{n-1} \frac{3\alpha^i}{2(2^i)}}) \tag{2.16}$$

for all $x \in X$ and $a > 0$. Taking the limit as $n \to \infty$ and using (N_6) we get

$$N(f(2x) - 8f(x) - A(x), a) \geq N''_1(x, \frac{a(2-\alpha)}{3})$$

for all $x \in X$ and $a > 0$. Now, we show that A is additive. Since $\lim_{n\to\infty} \frac{a(2^n)}{2\alpha^n} = \infty,$ we obtain

$\lim_{n\to\infty} N''_1(x, \frac{a(2^n)}{2\alpha^n}) = 1$. So, by (2.13) we obtain

$$A(2x) = 2A(x) \tag{2.17}$$

for all $x \in X$. On the other hand we have

$$N(DA(x,y),a) = \lim_{n\to\infty} N(\frac{1}{2^n} Dg(2^n x, 2^n y), a)$$

$$= \lim_{n\to\infty} N(\frac{1}{2^n}[Df(2^{n+1}x, 2^{n+1}y) - 8Df(2^n x, 2^n y)], a)$$

$$\geq \lim_{n\to\infty} \min\{N'(\varphi_1(2^{n+1}x, 2^{n+1}y), \frac{2^n a}{2}), N'(\varphi_1(2^n x, 2^n y), \frac{2^n a}{16})\} = 1$$

for all $x, y \in X$ and all $a > 0$. Hence the mapping *A* satisfies (1.2). So by the Lemma 2.1 of (Khodaei, 2010), the mapping $x \mapsto A(2x) - 8A(x)$ is additive. Therefore (2.17) implies that the mapping *A* is additive. To prove the uniqueness of *A*, let $A' : X \to Y$ be another additive mapping satisfying (2.3). Fix $x \in X$. Clearly $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $n \in N$. It follows from (2.3) that

$$N(A(x) - A'(x), a) = N(\frac{A(2^n x)}{2^n} - \frac{A'(2^n x)}{2^n}, a)$$

$$= \min\{N(\frac{A(2^n x)}{2^n} - \frac{g(2^n x)}{2^n}, \frac{a}{2}), N(\frac{g(2^n x)}{2^n} - \frac{A'(2^n x)}{2^n}, \frac{a}{2})\}$$

$$\geq N''(2^n x, \frac{a(2^n)(2-\alpha)}{6}) \geq N''_1(x, \frac{a(2^n)(2-\alpha)}{6\alpha^n})$$

for all $x \in X$ and $a > 0$.

Since $\lim_{n\to\infty} \frac{a(2^n)(2-\alpha)}{6\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N''_1(x, \frac{a(2^n)(2-\alpha)}{6\alpha^n}) = 1$. Therefore, $N(A(x) - A'(x), a) = 1$ for all $x \in X$ and $a > 0$, whence $A(x) = A'(x)$.

Theorem 2.2. Let $\varphi_2 : X \times X \to Z$ be a function such that for some $\alpha > 2$

$$N'(\varphi_2(\frac{x}{2}, \frac{y}{2}), a) \geq N'(\varphi_2(x, y), \alpha a)$$

for all $x \in X, y \in \{x, 2x\}$ and $a > 0$, and $\lim_{n\to\infty} N'(2^n \varphi_2(2^{-n}x, 2^{-n}y), a) = 1$ for all $x, y \in X$ and all $a > 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfying (2.2) for all $a > 0$ and all $x, y \in X$. Then the limit

$$A(x) = N - \lim_{n\to\infty} 2^n [f(\frac{x}{2^{n-1}}) - 8f(\frac{x}{2^n})]$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying

$$N(f(2x) - 8f(x) - A(x), a) \geq N''_2(x, \frac{a(\alpha - 2)}{3})$$

for all $x \in X$ and all $a > 0$,

$$N''_2(x, a) := \min\{N'(\varphi_2(0, x), a), N'(\varphi_2(x, x), a), N'(\varphi_2(x, 2x), a)\}.$$

Proof. The techniques are completely similar to those of Theorem 2.1.

Theorem 2.3. Let $\varphi_3 : X \times X \to Z$ be a function such that for some 0<α<8.

$$N'(\varphi_3(2x, 2y), a) \geq N'(\alpha \varphi_3(x, y), a) \tag{2.18}$$

for all $x \in X, y \in \{x, 2x\}$ and all $a > 0$, and $\lim_{n\to\infty} N'(\varphi_3(2^n x, 2^n y), 8^n a) = 1$ for all $x, y \in X$ and $a > 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (2.2) for all $a > 0$ and $x, y \in X$. Then the limit

$$C(x) = N - \lim_{n\to\infty} \frac{1}{8^n} [f(2^{n+1}x - 2f(2^n x)]$$

exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying

$$N(f(2x) - 2f(x) - C(x), a) \geq N''_3(x, \frac{a(8 - \alpha)}{3}) \tag{2.19}$$

for all $x \in X$ and all $a > 0$, where

$$N''_3(x, a) := \min\{N'(\varphi_3(0, x), a), N'(\varphi_3(x, x), a), N'(\varphi_3(x, 2x), a)\} \tag{2.20}$$

Proof. Similar to the proof of Theorem 2.1, we have

$$N(f(4x) - 10f(2x) + 16f(x), 3a) \geq N''_3(x, a) \tag{2.21}$$

for all $x \in X$. Let $h : X \to Y$ be a mapping defined by $h(x) = f(2x) - 2f(x)$, then (2.21) means

$$N(h(2x) - 8h(x), 3a) \geq N''_3(x, a) \tag{2.22}$$

for all $x \in X$ and all $a > 0$. Replacing *x* by $2^n x$ in (2.22), we get

$$N(h(2^{n+1}x) - 8h(2^n x), 3a) \geq N''_3(2^n x, a) \tag{2.23}$$

for all $x \in X$ and $a > 0$. Using (2.18) we get

$$N(h(2^{n+1}x) - 8h(2^n x), 3a) \geq N''_3(x, \alpha^{-n} a) \tag{2.24}$$

for all $x \in X$ and. $a > 0$ .Replacing α by $\alpha^n a$ we see that

$$N(\frac{h(2^{n+1}x)}{8^{n+1}} - \frac{h(2^n x)}{8^n}, \frac{3a\alpha^n}{8(8^n)} \geq N''_3(x, a) \tag{2.25}$$

for all $x \in X$ and $a > 0$. It follows from $\frac{h(2^n x)}{8^n} - h(x) = \sum_{i=0}^{n-1} \frac{h(2^{i+1}x)}{8^{i+1}} - \frac{h(2^i x)}{8^i}$ and (2.25) that

$$N(\frac{h(2^n x)}{8^n} - h(x), \sum_{i=0}^{n-1} \frac{3a\alpha^i}{8(8^i)} \geq \min \bigcup_{i=0}^{n-1} \{N(\frac{h(2^{i+1}x)}{8^{i+1}} - \frac{h(2^i x)}{8^i}, \frac{3a\alpha^i}{8(8^i)})\} \geq N''_3(x, a) \tag{2.26}$$

for all $x \in X$ and $a > 0$. Replacing *x* with $2^m x$ in (2.26) we observe that

$$N(\frac{h(2^{n+m}x)}{8^{n+m}} - \frac{h(2^m x)}{8^m}, \sum_{i=0}^{n-1} \frac{3a\alpha^i}{8(8^{i+m})}) \geq N''_3(2^m x, a) \geq N''_3(x, \frac{a}{\alpha^m}),$$

Whence

$$N(\frac{h(2^{n+m}x)}{8^{n+m}} - \frac{h(2^m x)}{8^m}, \sum_{i=m}^{n+m-1} \frac{3a\alpha^i}{8(8^i)}) \geq N''_3(x, a)$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$.
Hence

$$N(\frac{h(2^{n+m}x)}{8^{n+m}} - \frac{h(2^m x)}{8^m}, a) \geq N''_3(x, \frac{a}{\sum_{i=m}^{n+m-1} \frac{3\alpha^i}{8(8^i)}}) \tag{2.27}$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$.Since $0 < \alpha < 8$and $\sum_{i=0}^{\infty} (\frac{\alpha}{8})^i < \infty$ the Cauchy criterion for convergence and (N_5) imply that $\{\frac{h(2^n x)}{8^n}\}$ is a Cauchy sequence in (Y.N).

Since (Y.N) is a fuzzy Banach space, this sequence converges to some point $C(X) \in Y$.

So one can define the mapping $C: X \to Y$ by $C(x) := n-\lim_{n\to\infty} \frac{h(2^n x)}{8^n}$ for all $x \in X$.

Letting $m = 0$ in (2.27), we get

$$N(\frac{h(2^n x)}{8^n} - h(x), a) \geq N''_3(x, \frac{a}{\sum_{i=0}^{n-1} \frac{3\alpha^i}{8(8^i)}}) \quad (2.28)$$

for all $x \in X$ and $a > 0$. Taking the limit as $n \to \infty$ and using (N_6) we get

$$N(f(2x) - 2f(x) - C(x), a) \geq N''_3(x, \frac{a(8-a)}{3})$$

for all $x \in X$ and $a > 0$. Now, we show that C is cubic. Since $\lim_{n\to\infty} \frac{a(2^n)}{8\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N''_3(x, \frac{a(2^n)}{8\alpha^n}) = 1$

So, by (2.25) we obtain

$$C(2x) = 8C(x) \quad (2.29)$$

for all $x \in X$. On the other hand we have

$$N(DC(x,y), a) = \lim_{n\to\infty} N(\frac{1}{8^n} Dh(2^n x, 2^n y), a)$$

$$= \lim_{n\to\infty} N(\frac{1}{8^n}[Df(2^{n+1}x, 2^{n+1}y) - 2Df(2^n x, 2^n y)], a)$$

$$\geq \lim_{n\to\infty} \min\{N'(\varphi_3(2^{n+1}x, 2^{n+1}y), \frac{8^n a}{2}), N'(\varphi_3(2^n x, 2^n y), \frac{8^n a}{4})\} = 1$$

for all $x, y \in X$. And all $a > 0$. Hence the mapping C satisfies (1.2). So by the Lemma 2.2 of , the mapping $x \mapsto C(2x) - 2C(x)$ is cubic. Therefore (2.29) implies that the mapping C is cubic. To prove the uniqueness of C, let $C': X \to Y$ be another additive mapping satisfying (2.19). Fix $x \in X$. Clearly $C(2^n x) = 8^n C(x)$ and $C'(2^n x) = 8^n C'(x)$ for all $n \in N$. It follows from (2.19) that

$$N(C(x) - C'(x), a) = N(\frac{C(2^n x)}{8^n} - \frac{C'(2^n x)}{8^n}, a)$$

$$\geq \min\{N(\frac{C(2^n x)}{2^n} - \frac{h(2^n x)}{8^n}, \frac{a}{2}), N(\frac{h(2^n x)}{8^n} - \frac{C'(2^n x)}{8^n}, \frac{a}{2})\}$$

$$\geq N''_3(2^n x, \frac{a(8^n)(8-\alpha)}{6}) \geq N''_3(x, \frac{a(8^n)(8-\alpha)}{6\alpha^n})$$

for all $x \in X$ and $a > 0$.

Since $\lim_{n\to\infty} \frac{a(8^n)(8-\alpha)}{6\alpha^n} = \infty$, Now, we show that C is cubic.

Since $\lim_{n\to\infty} N''_3(x, \frac{a(8^n)(8-\alpha)}{6\alpha^n}) = 1$. we obtain $\lim_{n\to\infty} N''_3(x, \frac{a(8^n)(8-a)}{6\alpha^n}) = 1$. Therefore,

$N(C(x) - C'(x), a) = 1$ for all $x \in X$ and $a > 0$, whence $C(x) = C'(x)$.

Theorem 2.4. Let $\varphi_4 : X \times X \to Z$ be a function such that for some $\alpha > 8$

$$N'(\varphi_4(\frac{x}{2}, \frac{y}{2}), a) \geq N'(\varphi_4(x, y), \alpha a)$$

for all $x \in X$, $y \in \{x, 2x\}$ and $a > 0$, and $\lim_{n\to\infty} N'(8^n \varphi_4(2^{-n}x, 2^{-n}y), a) = 1$ for all $x, y \in X$ and $a > 0$.

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfying (2.2) for all $a > 0$ and all $x, y \in X$. Then the limit

$$C(x) = N - \lim_{n\to\infty} 8^n [f(\frac{x}{2^{n-1}}) - 2f(\frac{x}{2^n})]$$

exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique additive mapping satisfying

$$N(f(2x) - 2f(x) - C(x), a) \geq N''_4(x, \frac{a(\alpha - 8)}{3})$$

for all $x \in X$ and $a > 0$, where

$$N''_4(x, a) := \min\{N'(\varphi_4(0, x), a), N'(\varphi_4(x, x), a), N'(\varphi_4(x, 2x), a).\}$$

Proof. The techniques are completely similar to those of Theorem 2.3.
We now prove our main theorem in this section.

Theorem 2.5. Let $\varphi : X \times X \to Z$ be a function such that for some $0 < \alpha < 2$

$$N'(\varphi(2x, 2y), a) \geq N'(\alpha\varphi(x, y), a)$$

for all $x \in X$, $y \in \{x, 2x\}$ and $a > 0$, and $\lim_{n\to\infty} N'(\varphi(2^n x, 2^n y), 2^n a) = 1$ for all $x, y \in X$ and $a > 0$

Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$N(Df(x,y),a) \geq N'(\varphi(x,y),a)$$

for all $a > 0$ and all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$N(f(x) - A(x) - C(x), a) \geq N''(x,a) \tag{2.30}$$

for all $x \in X$ and all $a > 0$, where

$$N''(x,a) := \min\{N''_1(x, a(2-\alpha)), N''_3(x, a(8-\alpha))\},$$

$$N''_1(x,a) := \min\{N'(\varphi_1(0,x),a), N'(\varphi_1(x,x),a), N'(\varphi_1(x,2x),a)\},$$

$$N''_3(x,a) := \min\{N'(\varphi_3(0,x),a), N'(\varphi_3(x,x),a), N'(\varphi_3(x,2x),a)\},$$

Proof. By Theorems 2.1 and 2.3, there exists an additive mapping $A_1 : X \to Y$ and a cubic mapping $C_1 : X \to Y$ such that

$$N(f(2x) - 8f(x) - A_1(x), a) \geq N''_1(x, \frac{a(2-\alpha)}{3})$$

$$N(f(2x) - 2f(x) - C_1(x), a) \geq N''_2(x, \frac{a(8-\alpha)}{3})$$

for all $x \in X$ and all $a > 0$. Therefore it follows from the last inequalities that

$$N(f(x) + \frac{1}{6}A_1(x) - \frac{1}{6}C_1(x), a) \geq \min\{N''_1(x, a(2-\alpha)), N''_3(x, a(8-\alpha))\}$$

for all $x \in X$ and all $a > 0$. So we obtain (2.30) by letting $A(x) = \frac{-1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$ for all $x \in X$.

To prove the uniqueness of *A* and *B*, let $A_0, C_0 : X \to Y$ be another additive and cubic mappings satisfying (2.30).

Let $A' = A - A_0$ and $C' = C - C_0$. So

$$N(A'(2^n x) + C'(2^n x), a) \geq \min\{N(f(2^n x) - A(2^n x) - C(2^n x), \frac{a}{2}),$$

$$N(f(2^n x) - A_0(2^n x) - C_0(2^n x), \frac{a}{2})\} \geq N''(x, \frac{a}{2\alpha^n}) \tag{2.31}$$

for all $x \in X$ and all $a > 0$. Thus

$$N(\frac{1}{8^n}[A'(2^n x) + C'(2^n x)], a) \geq N''(x, \frac{8^n a}{2\alpha^n})$$

for all $x \in X$ and all $a > 0$. Since $\lim_{n\to\infty} \frac{8^n a}{2\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N''(x, \frac{8^n a}{2\alpha^n}) = 1$. Therefore

$\lim_{n\to\infty} N(\frac{1}{8^n}[A'(2^n x) + C'(2^n x)], a) = 1$. So, $\lim_{n\to\infty} \frac{1}{8^n}[A'(2^n x) + C'(2^n x)] = 0$.

Hence $C' = 0$. It follows from (2.31) and (2.3) that $N(A'(x), a) \geq N''_1(x, \frac{a(2-\alpha)}{6})$ for all $x \in X$ and all $a > 0$. Therefore $A' = 0$.

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