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A New Approach Based on Homotopy Analysis Method for Solving System of Linear Equations

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Abstract: In this paper, we present an efficient numerical algorithm for solving system of linear equations based on homotopy analysis method. Some numerical illustrations are given to show the efficiency of the proposed method.

Key words: Homotopy analysis method, System of linear equations.

INTRODUCTION

Approximating the solutions of the system of linear and nonlinear equations has widespread applications in applied mathematics. Liao, (1992) employed the basic ideas of homotopy to propose a general method for nonlinear problems, and modified it step by step (Liao, 1995; 1997; 1999; 2004; 2005). This method has been successfully applied to solve many types of nonlinear problems. Following Liao, an analytic approach based on the same theory in 1998, which is so called "homotopy perturbation method" (HPM), is provided by He (1999; 2000; 2003; 2005). In this article, "homotopy analysis method" HAM is applied to the solution of the system Ax = b and the convergence of the method is considered under certain conditions.

2 Analysis of the method:

Consider the system

$$Ax = b, \tag{2.1}$$

where

$A=[a_{ij}]$, $x=[x_j]$, $b=[b_i]$, $i=1,2,...,\quad j=1,2,...$

let

$N(u) = Au - b, \quad L(u) = u,$

Let q $\in[0, 1]$ denotes an embedding parameter, $\hbar \neq 0$ an auxiliary parameter, $H(u) \neq 0$ an auxiliary function, and ` an auxiliary linear operator. We construct the zero-order deformation equation Liao, (1997)

$$(1-q)L[v(q)-u_0] = \hbar H(u)q\{A(v(q))-b\} \tag{2.2}$$

where u_0 is the initial approximation of u, and $v(q)$ is a unknown function. It should be emphasized that one has great freedom to choose the initial guess value, the auxiliary linear operator, the auxiliary parameter $\hbar$, and the auxiliary function $H(u)$. Obviously, when q=0 and q=1, it holds

$v(0) = u_0, \qquad v(1) = u$

respectively. When q increases from 0 to 1, $v(q)$ varies from the initial guess u_0 to the solution u. Expanding $v(q)$ in Taylor series with respect to the embedding parameter q, one has

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$$v(q) = u_0 + \sum_{m=1}^{\infty} u_m q^m, \tag{2.3}$$

where

$$u_m = \frac{1}{m!} \frac{d_m v(q)}{dq^m}\Big|_{q=0}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen that the series (2.5) converges at q = 1, one has

$$u = u_0 + \sum_{m=1}^{\infty} u_m q^m, \tag{2.4}$$

which must be one of the solutions of (2.1), as proved by Liao (2003). It is very important to ensure the convergence of series (2.5) at q = 1, otherwise, the series (2.4) has no meanings. $\hbar = -1$ and $H(u)$ = 1, we obtained "homotopy perturbation method" (He, 2004; Liao, 1994). Setting $L \equiv L$ and $H(u) \equiv 1$, we have the high-order deformation equation Liao, (1997)

$$L\left[u(m) - X_m u_{m-1}\right] = \hbar R_m(u_0, u_1, ..., u_m)$$

where

$$Xm = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1 \end{cases}$$

$$R_m(u_0, u_1, ..., u_m) = \frac{1}{(m-1)!} \frac{d^m \{A(v(q)) - b\}}{dq^m}\Big|_{q=0}$$

if we take u0 = 0 then we have

$$\begin{aligned} u_1 &= -\hbar b, \\ u_2 &= (\hbar A + I)u_1 \\ &= -\hbar(\hbar A + I)b, \\ u_3 &= (\hbar A + I)u_2 \\ &= -\hbar(\hbar A + I)^2 b, \end{aligned}$$

and in general

$$u_{m+1} = (\hbar A + I)u_n, \quad n = 1, 2, ...$$

hence, the solution can be of the form

$$u = u_0 + u_1 + u_2 + ...,$$

or

$$u = -\hbar[I + (\hbar A + I) + (\hbar A + I)^2 + ...]b.$$

Theorem 1.

The sequence

$$u^{[m]} = [-\hbar \sum_{k=0}^{m} (\hbar A + I)^k,]b,$$

is a Chauchy sequence if

$$\|\hbar A + I\| < 1$$

Proof. we must show that lim

$$\lim_{m\to\infty}\left\|u^{[m+p]} - u^{[m]}\right\| = 0$$

so for showing this we can write

$$u^{[m+p]} - u^{[m]} = \left[-\hbar\sum_{k=1}^{p}(\hbar A + I)^{m+k}\right]b$$

or

$$\left\|u^{[m+p]} - u^{[m]}\right\| \le \|-\hbar\|\sum_{k=1}^{p}\left\|(\hbar A + I)^{m+k}\right\|$$

lrt $\gamma = \left\|(\hbar A + I)^{m+k}\right\|$, then

$$\left\|u^{[m+p]} - u^{[m]}\right\| \le \|-\hbar\|\gamma^m\sum_{k=1}^{p}\gamma^k \le \|-\hbar b\|\left(\frac{\gamma^p - 1}{\gamma - 1}\right)\gamma^m,$$

now if $\gamma < 1$, then we have

$$\lim_{m\to\infty}\left\|u^{[m+p]} - u^{[m]}\right\| \le \|-\hbar b\|\left(\frac{\gamma^p - 1}{\gamma - 1}\right)\left(\lim_{M\to\infty}\gamma^m\right)$$

hence, we obtain

$$\lim_{m\to\infty}\left\|u^{[m+p]} - u^{[m]}\right\| = 0$$

which completes the proof. □

3 Numerical results:

Example 1:

Approximate the solution of the system

$$\begin{cases} 4x + y - z = 7 \\ -x + 6y + 2z = 9 \\ y - 3z = 5 \end{cases}$$

The true solution is u = (1, 2, -1) . For the given system we have

Au=b

where

$$A = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 6 & 2 \\ 0 & 1 & -3 \end{bmatrix}. \begin{cases} b^t = [7, 9, 5], \\ u^t = [x, y, z], \end{cases}$$

since *A* is diagonally dominated we write the new system as follows

$Bu = q$,

for which

$$B = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}, \quad q^t = \left[\frac{7}{4}, \frac{9}{6}, \frac{5}{3}\right]$$

from (2.5) we have

$$u = -\hbar[I + (-\hbar B + I) + (\hbar B + I)^2 + \ldots]q,$$

and using six terms, we approximate the series solution when $\hbar = 0.79$

$$u \approx u_0 + u_1 + u_2 + u_3 + u_4 + u_5,$$

$u^t \approx$ [0.987735, 2.00064, -0.988711].

Example 2:

Solve the system $Au = b$, where

$$A = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}, \quad b = \begin{bmatrix} 1.2 \\ 0.5 \\ 0.5 \end{bmatrix}$$

The true solution is $u = (1, 1, 1)$.
the ten-term approximation when $\hbar = -2.1$ is

$$u = \begin{bmatrix} 1.016615790 \\ 0.995847290 \\ 0.995847290 \end{bmatrix}$$

Conclusion:

In this paper, we used homotopy analysis method to approximate the solution of system of linear equations in terms of the subtraction of the coefficient and unit matrices. Solved problems show the convergence of the method increases as the coefficient matrix becomes more strictly diagonally dominated.

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