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Minimax Estimation of the Parameter of the Burr Type Xii Distribution

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Abstract: In this paper the classical estimators of the shape parameter θ for the Burr Type XII distribution, such as, the Maximum Likelihood Estimator (MLE), the Uniformly Minimum Variance Unbiased Estimator (UMVUE), and the Minimum Mean Squared Error (MinMSE) estimator are obtained. Then the problem of finding the minimax estimators of this parameter under the squared log error, precautionary, and weighted balanced squared error loss functions by applying the theorem of Lehmann [1950] is concerned. The obtained results have been interpreted in the light of two-person zero sum game. All these estimators are compared empirically using Monte Carlo simulation.

Key words: Minimax estimator, Burr type XII distribution, Precautionary loss function, Weighted Balanced type loss function, Game theory, Monte-Carlo simulation.

INTRODUCTION

The minimax estimation is an upgraded non-classical approach in the estimation area of statistical inference, which was introduced by Wald (1950) from the concept of game theory. It opens a new dimension in statistical estimation and enriches the method of point estimations. Von Neumann (1944) introduced the word minimax in game theory which is the optimum strategy of the second player in the two person zero game. According to Wald (1945), "mimimax approach tries to guard against the worst by requiring that the chosen decision rule should provide maximum protection against the highest possible risk". An estimator having this property is called a minimax estimator.

The Burr system of distributions was constructed by Burr (1942). The Burr-XII distribution is quite flexible as a lifetime model. The flexibility of the distribution arises from the fact that it has a non-monotone hazard function which makes it appropriate for representing the lifetime for many products, Soliman (2002). The probability density function of a Burr-XII distributed random variable is given by

$$f(x;\lambda,\theta)=\frac{\theta\lambda x^{\lambda-1}}{(1+x^{\lambda})^{\theta+1}}, \quad x>0, \quad \lambda,\theta>0 \quad , \tag{1.1}$$

where θ and λ are shape and scale parameters, respectively. If $\lambda\le 1$, $f(x;\lambda,\theta)$ is strictly decreasing. On the other hand, if $\lambda>1$, $f(x;\lambda,\theta)$ increases on $(0,x]$ and then decreases on $[x,\infty)$ where $x=\left[(\lambda-1)/(\theta\lambda+1)\right]^{1/\lambda}$. The distribution function $(c.d.f)$ is;

$$F(x;\lambda,\theta)=1-(1+x^{\lambda})^{-\theta}, \quad x>0, \quad \lambda,\theta>0 \quad . \tag{1.2}$$

The density functions of the Burr-XII distribution can take different shapes. For λ=1 and different values of θ it is a decreasing function and for λ=2 and different values of θ it is a unimodal, skewed, right.

Podder et al. (2004) studied the minimax estimator of the parameter of the Pareto distribution under Quadratic and MLINEX loss functions. Also, Dey (2008) studied the minimax estimator of the parameter for the Rayleigh distribution under quadratic loss function. Shadrokh and Pazira (2010) studied the minimax estimator of the parameter for the minimax distribution under several loss functions. In this paper, we shall estimate the parameter θ (when λ is known) by using the technique of minimax approach which is essentially a Bayesian approach. The most important element in the minimax approach is the specification of a distribution function on the parameter space, which is called prior distribution. In addition to the prior distribution, the minimax estimator for a particular model depends strongly on the loss function assumed. The basic difference between the philosophy of the minimax and classical estimation is that the parameter of the

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distribution is assumed to be a random variable, in the first approach, whereas a fixed point in the second one for small sample size. It has been observed that the non-classical minimax approach is better than the classical approach. In many of the non-classical estimations the symmetrical loss functions are considered. But there are some real life situations where the use of the symmetrical loss functions may be inappropriate. In some cases a given positive error may be more serious than a given negative error and vice-versa.

This paper is organized as follows: In Section 2, we find the classical estimators of the shape parameter θ for the Burr-XII distribution. The minimax estimators of θ under three type of loss functions, i.e., Squared Log Error (which is symmetric), Precautionary (which is asymmetric), and Weighted Balanced Squared Error are obtained in section 3. In Section 4, we interpret the minimax estimators with two-person zero-sum game. In Section 5, a simulation study is carried out to compare these estimators. The simulation results and discussions are provided in Section 6.

2. Classical Estimations:

In this section, we obtain the classical estimators of the parameter θ for the Burr-XII distribution and compare these estimators based on their mean squared errors (MSE's).

Let $X_1, X_2, ..., X_n$ be a random sample from density (1.2) (when λ is known). The likelihood function is given by

$$L(\theta) \propto \theta^n \left[\prod_{i=1}^{n} (1+x_i^{\lambda}) \right]^{-(\theta-1)} , \tag{2.1}$$

then the log-likelihood function is

$$\ell(\theta) \propto n \ln\theta - (\theta-1) \sum_{i=1}^{n} \ln(1+x_i^{\lambda}) , \tag{2.2}$$

hence,

$$\frac{\partial \ell(\theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln(1+x_i^{\lambda}) = 0 , \tag{2.2}$$

thus the MLE of θ is

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln(1+X_i^{\lambda})} = \frac{n}{T} , \tag{2.3}$$

where $T = \sum_{i=1}^{n} \ln(1+X_i^{\lambda})$.

Here, we obtain the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of θ. Since the family of density (1.2) belongs to the exponential family, therefore, statistic T is a complete sufficient statistic for θ. It is easy to show that statistic T is distributed as gamma distribution with parameters n and θ, with the density

$g(t) = \theta^n \left(\Gamma(n)\right)^{-1} t^{n-1} e^{-\theta t} \; ; t > 0 , \theta > 0$. Thus

$$E_\theta(\frac{1}{T}) = \frac{\theta}{n-1} ,$$

hence, the UMVUE of θ is

$$\hat{\theta}_{UMVUE} = \frac{n-1}{T} . \tag{2.4}$$

We can find the Minimum Mean Squared Error (MinMSE) estimator in the class of estimators of the form $\frac{u}{T}$. Therefore

$$MSE_\theta(\frac{u}{T}) = E[(\frac{u}{T}-\theta)^2] = Var(\frac{u}{T}) + [E(\frac{u}{T})-\theta]^2 .$$

Whereas $E_\theta(T^r) = \frac{\Gamma(n+r)}{\theta^r\,\Gamma(n)},\ n+r>0$, thus

$$E(\frac{u}{T}) = u\,E(T^{-1}) = u\theta\frac{\Gamma(n-1)}{\Gamma(n)} = \frac{u\theta}{n-1}$$

and

$$Var(\frac{u}{T}) = u^2\,Var(T^{-1}) = \frac{u^2\,\theta^2}{(n-1)^2(n-2)}$$

then

$$MSE_\theta(\frac{u}{T}) = \theta^2\left[\frac{u^2}{(n-1)^2(n-2)} + \left(\frac{u}{(n-1)}-1\right)^2\right] \tag{2.5}$$

$$= r(u).$$

The derivative of $r(u)$ is

$$r'(u) = \theta^2\left[\frac{2u}{(n-1)^2(n-2)} + 2\left(\frac{u}{(n-1)}-1\right)\left(\frac{1}{(n-1)}\right)\right]$$

$$=0$$

that thereby u=n-1. Thus, the Min MSE estimator of θ is

$$\hat{\theta}_{MinMSE} = \frac{n-2}{T} \tag{2.6}$$

From (2.5), the MSE of the classical estimators of θ are calculated as follow:

$$MSE_\theta(\hat{\theta}_{MLE}) = \frac{(n+2)}{(n-1)(n-2)}\theta^2 ,$$

$$MSE_\theta(\hat{\theta}_{UMVUE}) = \frac{\theta^2}{n-2} ,$$

and

$$MSE_\theta(\hat{\theta}_{MinMSE}) = \frac{\theta^2}{n-1} .$$

It is easy to show that

$$MSE_\theta(\hat{\theta}_{MinMSE}) < MSE_\theta(\hat{\theta}_{UMVUE}) < MSE_\theta(\hat{\theta}_{MLE}) .$$

Minimax Estimations:

In this section, we obtain the minimax estimators of the parameter θ for the Burr-XII distribution. The derivation depends primarily on a theorem, which is due to Hodge and Lehmann (1950) and can be stated as follows:

3. Lehmann's Theorem:

Let $\tau = \{F_\theta;\theta\in\Theta\}$ be a family of distribution functions and D a class of estimators of θ. Suppose that

$d^* \in D$ is a Bayes estimator against a prior distribution $\xi^*(\theta)$ on the parameter space Θ, and the risk function $R(d^*, \theta) = constant$ on Θ; then d^* is a minimax estimator of θ.

The main results are contained in the following Theorems.

THEOREM 1:

Let $X = (X_1, X_2, ..., X_n)$ be n independently and identically distributed random variables drawn from the density (1.2). Then $\hat{\theta}_{MWBSE} = (n+2w-2) / \sum_{i=1}^{n} \ln(1+X_i^{\lambda})$ is the Minimax estimator of the parameter θ for the Weighted Balanced Squared Error type loss function given by:

$$L_{w,d_0}(\theta, d_1) = w\,q(\theta)(d_1 - d_0)^2 + (1-w)\,q(\theta)(d_1 - \theta)^2 \tag{3.1}$$

where $w \in [0,1)$, $q(\theta) = 1/\theta^2$, and d_0 is a "target" estimator, obtained from the criterion of maximum likelihood.

Proof:

To prove the theorem we have to use Lehmann's theorem and Theorem 1 of Jafari Jozani et al. (2006a). In order to prove the theorem it will be sufficient to show that $d_1 = (n+2w-2) / \sum_{i=1}^{n} \ln(1+x_i^{\lambda})$ is a minimax estimator of θ for the weighted balanced squared error loss function (3.1). This loss, which depends on the observed value of d_0, reflects a desire of closeness of d_1 both to: (i) θ in terms of weighted squared error loss, and (ii) the target estimator d_0 in terms of weighted squared distance. Jafari Jozani et al. (2006b) introduced an extended class of balanced type loss functions of the form given in (3.1), with $q(\cdot)$ being a suitable positive weight function. For the case of $q(\theta)=1$ and a least squares d_0 in (3.1) is equivalent to Zellner's (1994) balanced loss function. For the more details about this loss function see Jafari Jozani et al. (2006a).

If we can show that the risk of d_1 is constant, then the Theorem 1 will be proved. Since, $d_0 = n / \sum_{i=1}^{n} \ln(1+x_i^{\lambda})$ has constant risk $\frac{n+2}{(n-1)(n-2)}$ under $L_{0,d_0}(\theta, d_1) = (d_1/\theta - 1)^2$, Theorem 1 of Jafari Jozani et al. (2006a) by using the equation (2.6) tell us that $\frac{n}{T} - \frac{2(1-w)}{T} = \frac{n-2w-2}{T}$ is a minimax estimator with minimax risk $\frac{(1-w)^2}{n-1}\left(1 + \frac{w}{1-w} \cdot \frac{n+2}{n-2}\right)$, which is a constant w.r.t. θ.

So according to the Lehmann's theorem it follows that $d_1 = \hat{\theta}_{MWBSE} = \frac{(n+2w-2)}{\sum_{i=1}^{n} \ln(1+X_i^{\lambda})}$ is the minimax estimator of the parameter θ for the Burr-XII distribution under the Weighted Balanced Squared Error loss function of the form (3.1). Note that: for w=0 the minimax estimator $\hat{\theta}_{MWBSE}$ is identical to the classical estimator $\hat{\theta}_{MinMSE}$, *i.e.,* $\hat{\theta}_{MWBSE} = \hat{\theta}_{MinMSE}$ ■

THEOREM 2:

Let $X = (X_1, X_2, ..., X_n)$ be n independent and identically distributed random variables from the density (1.2).

Then $\hat{\theta}_{MSLE} = \exp\{\Gamma'(n)/\Gamma(n)\}\Big/\sum_{i=1}^{n}\ln(1+X_i^{\lambda})$ is the minimax estimator of the parameter θ for the Squared Log Error loss function of the type

$$L(\theta,d_2) = \left(\ln d_2 - \ln\theta\right)^2 \quad , \tag{3.2}$$

where d_2 is the estimate of θ.

Proof:

Using Lehmann's theorem, it will be sufficient to show that $d_2 = \exp\{\Gamma'(n)/\Gamma(n)\}\Big/\sum_{i=1}^{n}\ln(1+x_i^{\lambda})$ is a minimax estimator of θ for the symmetric loss function (3.2). Therefore, we have to find the Bayes estimator d_2 of θ. Then if we can show that the risk of d_2 is constant, then the Theorem 2 will be proved.

Let us assume that θ has Jeffrey's non-informative prior density defined as

$$g(\theta) \propto \frac{1}{\theta} \;\; ; \theta > 0 \quad . \tag{3.3}$$

Then the posterior distribution of θ for the given random sample $X = (X_1, X_2, ..., X_n)$ is

$$g(\theta \mid \underline{x}) = \frac{\theta^{n-1}}{\Gamma(n)} T^n e^{-\theta T} \quad ; \theta > 0, x > 0 \quad , \tag{3.4}$$

where $\mathrm{T} = \sum_{i=1}^{n}\ln(1+x_i^{\lambda})$. Which implies that $\theta \mid \underline{x}$ is distributed as Gamma distribution with parameters n and T and mean of the distribution is n/T.

Now the Bayes estimator of θ under the squared log error loss function (3.2) is

$$\hat{\theta}_{BSLE} = e^{E_{\theta}[\ln\theta]} \, ,$$

where

$$E_{\theta}[\ln\theta] = \int_{\theta}\left(\ln\theta\right) g(\theta \mid \underline{x})\, d\theta$$

$$= \frac{t^n}{\Gamma(n)}\int_0^{\infty}(\ln\theta)\,\theta^{n-1} e^{-t\theta}\, d\theta$$

Using the relation $\theta t = u \;\; \Rightarrow \;\; \theta = \frac{u}{t} \;\; \Rightarrow \;\; d\theta = \frac{1}{t} du$, we have

$$E_{\theta}\left[\ln\theta\right] = \frac{1}{\Gamma(n)}\int_0^{\infty}\ln\left(\frac{u}{t}\right) u^{n-1} e^{-u}\, du$$

$$= \frac{1}{\Gamma(n)}\int_0^{\infty}\left(\ln u - \ln t\right) u^{n-1} e^{-u}\, du$$

$$= \frac{1}{\Gamma(n)}\int_0^{\infty}\ln(u)\, u^{n-1} e^{-u}\, du - \frac{\ln t}{\Gamma(n)}\int_0^{\infty} u^{n-1} e^{-u}\, du$$

$$= \frac{\Gamma'(n)}{\Gamma(n)} - \ln t = \psi - \ln t$$

where ψ is the digamma function, *i.e.*, $\psi = \Gamma'(n)/\Gamma(n)$, and $\Gamma'(n) = \int_0^{\infty}\ln(u)\, e^{-u} u^{n-1}\, du$ is the first derivative of $\Gamma(n)$ with respect to n. Using this result we get

$$\hat{\theta}_{BSLE} = \exp\left\{\frac{\Gamma'(n)}{\Gamma(n)} - \ln t\right\} = \frac{e^{\psi}}{\mathrm{T}}$$

where $\mathrm{T} = \sum_{i=1}^{n} \ln(1 + x_i^{\lambda})$.

Now the risk function under the squared log error loss function (3.2) is given by

$$R_{SLE}(\theta) = E\left[L(\hat{\theta}_{BSLE}, \theta) \right]$$

$$= E\left[\left(\ln \hat{\theta}_{BSLE} - \ln \theta \right)^2 \right]$$

$$= E\left[\left(\ln \hat{\theta}_{BSLE} \right)^2 - 2(\ln \theta)(\ln \hat{\theta}_{BSLE}) + \left(\ln \theta \right)^2 \right]$$

$$= E\left[\left(\psi - \ln \mathrm{T} \right)^2 - 2(\ln \theta)(\psi - \ln \mathrm{T}) + \left(\ln \theta \right)^2 \right]$$

$$= E\left[\left(\psi - \ln \mathrm{T} \right)^2 \right] - 2(\ln \theta) E\left[\psi - \ln \mathrm{T} \right] + \left(\ln \theta \right)^2$$

$$= \psi^2 - 2\psi\, E\left[\ln \mathrm{T} \right] + E\left[\left(\ln \mathrm{T} \right)^2 \right] - 2\psi \ln \theta + 2(\ln \theta) E\left[\ln \mathrm{T} \right] + \left(\ln \theta \right)^2 .$$

Hence we get,

$$E\left[\ln \mathrm{T} \right] = \frac{\theta^n}{\Gamma(n)} \int_0^\infty (\ln t)\, e^{-\theta t}\, t^{n-1}\, dt \quad ,$$

using the relation $\theta t = y \;\Rightarrow\; t = \frac{1}{\theta} y \;\Rightarrow\; dt = \frac{1}{\theta} dy$ we have

$$E(\ln \mathrm{T}) = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \ln(\frac{1}{\theta} y)\, e^{-y}\, (\frac{1}{\theta} y)^{n-1} \frac{1}{\theta} dy$$

$$= \frac{-\ln \theta}{\Gamma(n)} \int_0^\infty e^{-y}\, y^{n-1}\, dy + \frac{1}{\Gamma(n)} \int_0^\infty \ln y\, e^{-y}\, y^{n-1}\, dy$$

$$= -\ln \theta + \frac{\Gamma'(n)}{\Gamma(n)} = -\ln \theta + \psi. \tag{3.5}$$

Also we have,

$$E\left[(\ln \mathrm{T})^2 \right] = \frac{\theta^n}{\Gamma(n)} \int_0^\infty (\ln t)^2\, e^{-\theta t}\, t^{n-1}\, dt \quad ,$$

using the relation $\theta t = v \;\Rightarrow\; t = \frac{v}{\theta} \;\Rightarrow\; dt = \frac{1}{\theta} dv$ we obtain

$$E\left[(\ln \mathrm{T})^2 \right] = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \left(\ln(\frac{v}{\theta}) \right)^2 e^{-v}\, (\frac{1}{\theta} v)^{n-1} \frac{1}{\theta} dv$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty \left(\ln v - \ln \theta \right)^2 e^{-v}\, v^{n-1}\, dv$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty \left(\ln v \right)^2 e^{-v}\, v^{n-1}\, dv - \frac{2 \ln \theta}{\Gamma(n)} \int_0^\infty \left(\ln v \right) e^{-v}\, v^{n-1}\, dv + \frac{(\ln \theta)^2}{\Gamma(n)} \int_0^\infty e^{-v}\, v^{n-1}\, dv$$

$$= \zeta - 2\psi \ln \theta + (\ln \theta)^2$$

where $\zeta = \Gamma''(n)/\Gamma(n)$ and $\Gamma''(n) = \int_0^\infty (\ln v)^2\, e^{-v}\, v^{n-1}\, dv$ is the second derivative of $\Gamma(n)$ with respect to n. Using these results the risk function becomes

$$R_{SLE}(\theta) = \psi^2 - 2\psi(-\ln\theta + \psi) + \zeta - 2\psi\ln\theta + (\ln\theta)^2 - 2\psi\ln\theta + 2(\ln\theta)(-\ln\theta + \psi) + \left(\ln\theta\right)^2$$
$$= \zeta - \psi^2 = \Psi$$

where Ψ is the trigamma function, namely the first derivative of Ψ at n,which is a constant w.r.t. θ.

So from Lehmann's theorem it follows that $d_2 = \hat{\theta}_{MSLE} = \exp\{\Gamma'(n)/\Gamma(n)\} \Big/ \sum_{i=1}^{n} \ln(1+X_i^{\lambda})$ is the minimax estimator of the parameter θ of the Burr-XII distribution under the squared log error loss function (3.2).■

THEOREM 3:

Let $X = (X_1, X_2, ..., X_n)$ be n independently and identically distributed random variables drawn from the density (1.2). Then $\hat{\theta}_{MP} = \sqrt{n(n-1)} \,/\, \sum_{i=1}^{n} \ln(1+X_i^{\lambda})$ is the Minimax estimator of the parameter θ for the Precautionary loss function (see Norstrom (1996)) of the type

$$L(\theta, d_3) = \frac{(d_3 - \theta)^2}{d_3\,\theta} \tag{3.6}$$

where d_3 is the estimate of θ.

Proof:

Using Lehmann's theorem, it will be sufficient to show that $d_3 = \sqrt{n(n-1)} \,/\, \sum_{i=1}^{n} \ln(1+x_i^{\lambda})$ is a minimax estimator of θ for the asymmetric loss function (3.6). For this, first we have to find the Bayes estimator d_3 of θ. Then if we can show that the risk of d^3 is constant, then the proof is complete. Regarding to the Theorem 2 and using the non- informative prior, we get the posterior distribution of θ as equation (3.4).

Now the Bayes estimator of θ under the precautionary loss function (3.6) is

$$\hat{\theta}_{BMP} = \sqrt{\frac{E_{\theta}[\theta]}{E_{\theta}[1/\theta]}} \,.$$

Whereas,

$$E_{\theta}[\theta] = n \,/\, \mathrm{T} \,,$$

and

$$E_{\theta}[1/\theta] = \mathrm{T}/(n-1)$$

hence,

$$\hat{\theta}_{BMP} = \frac{\sqrt{n(n-1)}}{\sum_{i=1}^{n} \ln(1+X_i^{\lambda})} = \frac{\sqrt{n(n-1)}}{\mathrm{T}}$$

The risk function of the estimator $\hat{\theta}_{BMP}$ is

$$R_{MP}(\theta) = E\left[L(\hat{\theta}_{BMP}, \theta)\right]$$

$$= E\left[\frac{\hat{\theta}_{BMP}}{\theta} + \frac{\theta}{\hat{\theta}_{BMP}} - 2\right]$$

$$= \frac{1}{\theta} E\left[\hat{\theta}_{BMP}\right] + \theta E\left[\frac{1}{\hat{\theta}_{BMP}}\right] - 2$$

$$= \frac{\sqrt{n(n-1)}}{\theta} E\left[\frac{1}{\mathrm{T}}\right] + \frac{\theta}{\sqrt{n(n-1)}} E\left[\mathrm{T}\right] - 2 \quad .$$

Whereas,

$$E\left[\frac{1}{\mathrm{T}}\right] = \frac{\theta}{n-1} \quad \text{and} \quad E\left[\mathrm{T}\right] = \frac{n}{\theta} \ ,$$

therefore,

$$R_{MP}(\theta) = 2\left(\sqrt{\frac{n}{n-1}} - 1\right) \quad ,$$

which is a constant w.r.t. θ.

So from Lehmann's theorem it follows that $d_3 = \hat{\theta}_{MP} = \sqrt{n(n-1)} \Big/ \sum_{i=1}^{n} \ln(1 + X_i^{\lambda})$ is the minimax estimator of the parameter θ for the Burr-XII distribution under the precautionary loss function (3.6).■

4. Interpretation of Minimax Estimators with Two-Person Zero-Sum Game:

According to Wald (1950) the following statistical problem is equivalent to some two-person zero-sum-game between the statistician (Player-II) and nature (Player-I). Here the pure strategies of nature are the different values of θ in the interval $(0,\infty)$ and the mixed strategies of nature are the prior densities of θ in the interval $(0,\infty)$. The pure strategies of statistician are all possible decision functions in the interval $(0,\infty)$.

Expectation of the loss function $L(\theta,d)$ is the risk function, $R(\theta,d) = E\left[L(\theta,d)\right]$ which is the "gain" of player-I. $R(\xi,d)$ is the value of $\int R(\theta,d)\, d\xi(\theta)$, where $\xi(\theta)$ is the prior density θ. If the loss function is continuous in both d and θ and convex in d for each θ then there exist measures ξ^* and d^* for all θ and d so that the following relation holds.

$$R(\xi, d^*) \le R(\xi^*, d^*) \le R(\xi^*, d)$$

The number $R(\xi^*, d^*)$ is known to be the value of the game and ξ^* and d^* are the corresponding optimum strategies of Player-I and Player-II. In statistical terms ξ^* is the least favourable prior density of θ and d^* is a minimax estimator of θ. In fact, the value of the game is the loss of the statistician.

It has been shown that, here (I) $d_1^* = \hat{\theta}_{MWBSE} = (n + 2w - 2) \Big/ \sum_{i=1}^{n} \ln(1 + X_i^{\lambda})$ is the optimum strategy of player-II for the weighted balanced squared error loss function (3.1) and the value of the game is $R_{WBSE}(\xi^*, d_1^*) = \frac{(1-w)^2}{n-1}\left(1 + \frac{w}{1-w} \cdot \frac{n+2}{n-2}\right)$. (II) $d_2^* = \hat{\theta}_{MSLE} = \exp\{\Gamma'(n)/\Gamma(n)\} \Big/ \sum_{i=1}^{n} \ln(1 + X_i^{\lambda})$ is the optimum strategy of player-II for the squared log error loss function (3.2) and the value of the game is $R_{SLE}(\xi^*, d_2^*) = \frac{\Gamma''(n)}{\Gamma(n)} - \left(\frac{\Gamma'(n)}{\Gamma(n)}\right)^2$. (III) $d_3^* = \hat{\theta}_{MP} = \sqrt{n(n-1)} \Big/ \sum_{i=1}^{n} \ln(1 + X_i^{\lambda})$ is the optimum strategy of player-II for the precautionary loss function (3.6) and the value of the game is $R_{MP}(\xi^*, d_3^*) = 2\left(\sqrt{\frac{n}{n-1}} - 1\right)$. In all the cases $\xi^* = g(\theta) \propto \frac{1}{\theta}$; $\theta > 0$, is the optimum strategy for Player-I.

5. Simulation Study:

The estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{UMVUE}$ and $\hat{\theta}_{MinMSE}$ are classical estimators of the parameter θ for the Burr-XII distribution; whereas $\hat{\theta}_{MWBSE}$, $\hat{\theta}_{MSLE}$ and $\hat{\theta}_{MP}$ are minimax estimators under weighted balanced squared error, squared log error, and precautionary loss functions, respectively.

In section 2 we showed that, for the classical estimators, $MSE_\theta(\hat{\theta}_{MinMSE}) < MSE_\theta(\hat{\theta}_{UMVUE}) < MSE_\theta(\hat{\theta}_{MLE})$, but in this section our main aim is to compare the (classical and minimax) estimators in terms of their Biases and MSEs. The Biases and MSEs of the estimators are computed using the Monte-Carlo simulation study. The simulation is carried out for $\theta = 1$ *and* 2, and without loss of generality we take $\lambda = 2$, with sample size *n*=2,4,5,7,10(10)50. All results are based on 1000 replications. The obtained results are demonstrated in Tables 1 and 2 and also presented them in Figures 1 and 2.

From Table 1 and 2, we can see that between the classical estimators, estimator $\hat{\theta}_{MinMSE}$ is better than the rest, this is true for all values of *n*, see also Figure 1. From Table 1 and 2, we can see that between the minimax estimators, for all values of *n*, the minimax estimator under weighted balanced squared error loss function ($\hat{\theta}_{MWBSE}$) is better than the rest, see Figure 2. In general, estimator $\hat{\theta}_{MWBSE}$ is better than estimator $\hat{\theta}_{MinMSE}$ for small sample size (i.e for $n \le 10$). But, for large sample size ($n > 10$), the classical and minimax estimators ($\hat{\theta}_{MinMSE}$ and $\hat{\theta}_{MWBSE}$), are identical. Also we can see that between the minimax estimators, the minimax estimators under precautionary loss function ($\hat{\theta}_{MP}$) have the smallest estimated Biases as compared with the rest. On the other hand, the classical estimator $\hat{\theta}_{MLE}$, and the minimax estimators $\hat{\theta}_{MSLE}$, and $\hat{\theta}_{MP}$, are overestimation, but the rest are underestim ation.

Table 1: Biases and MSEs of different estimators for the parameter θ of the Burr-XII distribution when λ=2,θ=1 and *w*= 0.05 (MSE in parenthesis).

Sample Size	$\hat{\theta}_{MLE}$	$\hat{\theta}_{UMVUE}$	$\hat{\theta}_{MinMSE}$	$\hat{\theta}_{MWBSE}$	$\hat{\theta}_{MSLE}$	$\hat{\theta}_{MP}$
2	1.0395 (16.883)	0.0197 (3.9510)	-1.0000 (1.0000)	0.8980 (0.8459)	0.5564 (9.5118)	0. 4421 (8.0968)
4	0. 3454 (1.0128)	0. 0091 (0.5027)	-0. 3273 (0.3305)	-0. 2937 (0.3275)	0. 1812 (0.7215)	0. 1652 (0.6974)
5	0.2659 (0.6243)	0.0127 (0.3544)	-0.2405 (0.2591)	-0. 2152 (0.2571)	0.1416 (0.4703)	0.1322 (0.4603)
7	0.1455 (0.2564)	0.0182 (0.1731)	-0.1818 (0.1530)	-0.1654 (0.1522)	0.0647 (0.2074)	0.0605 (0.2053)
10	0.0871 (0.1348)	-0.0216 (0.1035)	-0.1303 (0.0984)	-0.1194 (0.0977)	0.0332 (0.1160)	0.0313 (0.1154)

Table 1: Continue

20	0.0524 (0.0626)	-0.0002 (0.0540)	-0.0528 (0.0512)	-0.0475 (0.0513)	0.0262 (0.0576)	0.0258 (0.0575)
30	0.0314 (0.0378)	-0.0030 (0.0344)	-0.0374 (0.0334)	-0.0340 (0.0334)	0.0142 (0.0358)	0.0140 (0.0358)
40	0.0305 (0.0300)	0.0047 (0.0276)	-0.0210 (0.0266)	-0.0184 (0.0266)	0.0176 (0.0286)	0.0175 (0.0286)
50	0.0230 (0.0211)	0.0025 (0.0198)	-0.0179 (0.0193)	-0.0159 (0.0193)	0.0128 (0.0203)	0.0127 (0.0203)

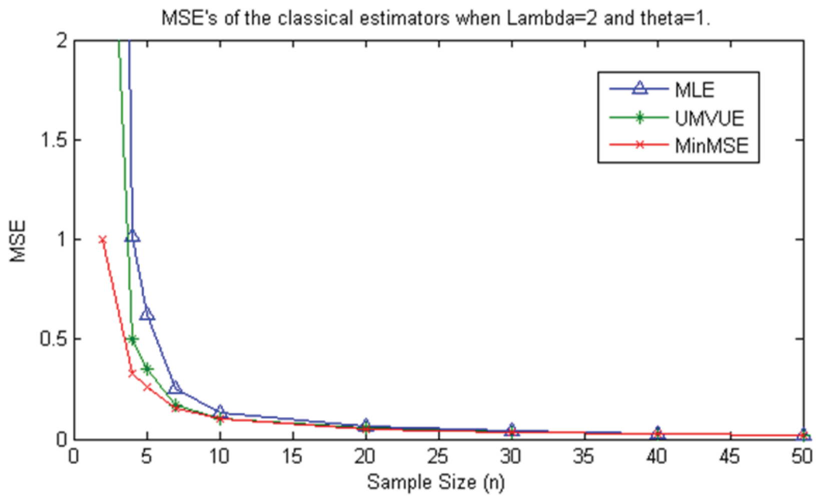

Fig. 1: MSE's of the classical estimators, for different value of n.

Table 2: Biases and MSEs of different estimators for the parameter θ of the Burr-XII distribution when $\lambda=2,\theta=1$ and $w= 0.05$ (MSE in parenthesis).

Sample Size	$\hat{\theta}_{MLE}$	$\hat{\theta}_{UMVUE}$	$\hat{\theta}_{MinMSE}$	$\hat{\theta}_{MWBSE}$	$\hat{\theta}_{MSLE}$	$\hat{\theta}_{MP}$
2	2.0792 (67.533)	0.0396 (15.804)	-2.0000 (4.0000)	-1.7960 (3.3838)	1.1128 (38.047)	0.8844 (32.387)
4	0.6908 (4.0513)	0.0181 (2.0107)	-0.6546 (1.3220)	-0.5873 (1.3201)	0.3624 (2.8861)	0.3303 (2.7897)
5	0.5317 (2.4970)	0.0254 (1.4178)	-0.4810 (1.0284)	-0.4303 (1.0263)	0.2832 (1.8811)	0.2645 (1.8413)
7	0.2910 (1.0255)	-0.0363 (0.6925)	-0.3636 (0.6122)	-0.3309 (0.6089)	0.1294 (0.8295)	0.1210 (0.8210)
10	0.1742 (0.5391)	-0.0432 (0.4139)	-0.2606 (0.3935)	-0.2389 (0.3908)	0.0665 (0.4640)	0.0627 (0.4618)
20	0.1049 (0.2503)	-0.0004 (0.2159)	-0.1056 (0.2050)	-0.0951 (0.2050)	0.0525 (0.2303)	0.0516 (0.2300)
30	0.0627 (0.1511)	-0.0060 (0.1375)	-0.0748 (0.1338)	-0.0679 (0.1337)	0.0285 (0.1431)	0.0281 (0.1430)
40	0.0610 (0.1198)	0.0095 (0.1105)	-0.0420 (0.1066)	-0.0369 (0.1067)	0.0353 (0.1145)	0.0351 (0.1144)
50	0.0460 (0.0845)	0.0051 (0.0791)	-0.0359 (0.0772)	-0.0318 (0.0772)	0.0255 (0.0814)	0.0254 (0.0814)

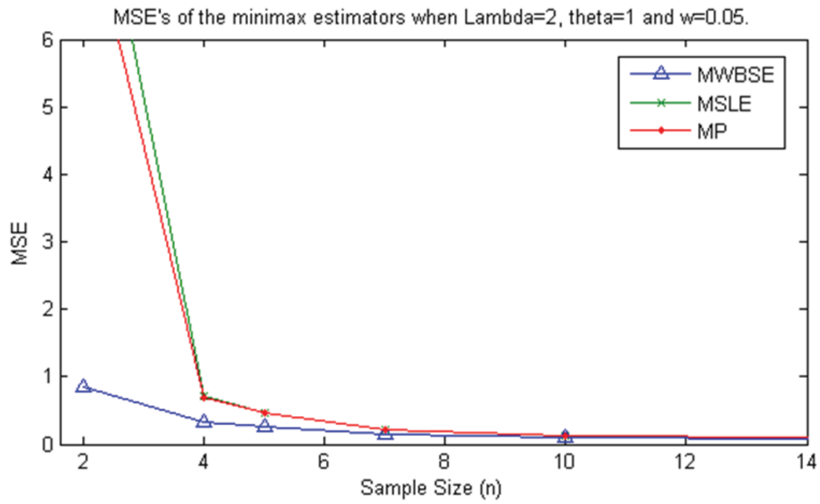

Fig. 2: MSE's of the minimax estimators, for small values of *n*.

6. Conclusion:

In this paper we obtained the Classical and Minimax estimators for the shape parameter of the Burr-XII distribution. We derived the minimax estimators under symmetric, asymmetric and balanced loss functions. We have shown that between the classical estimators, estimator $\hat{\theta}_{MinMSE}$ is better than the rest, and in between the minimax estimators, the minimax estimator under weighted balanced squared error loss function ($\hat{\theta}_{MWBSE}$) is better than the rest, these are true for all values of *n*. In general, in between this two estimators ($\hat{\theta}_{MinMSE}$ and $\hat{\theta}_{MWBSE}$), estimator $\hat{\theta}_{MWBSE}$ is better than estimator $\hat{\theta}_{MinMSE}$ for small sample size ($n \leq 10$). But, for large sample size ($n>10$), the classical and minimax estimators ($\hat{\theta}_{MinMSE}$ and $\hat{\theta}_{MWBSE}$), are identical. Thus, we suggest to use the minimax estimator under weighted balanced squared error loss function for estimating the shape parameter of the Burr-XII distribution.

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