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Two Successive Schemes for Numerical Solution of Linear Fuzzy Fredholm Integral Equations of the Second Kind

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Abstract: In this paper, two successive schemes for solving linear fuzzy Fredholm integral equations are presented. Using the parametric form of fuzzy numbers, we convert linear fuzzy Fredholm integral equation of the second kind to a linear system of integral equations of the second kind in the crisp case. We use two schemes, successive approximation and Taylor-successive approximation methods, to find the approximate solutions of the converted system, which are the approximate solutions for the fuzzy Fredholm integral equation of the second kind. The proposed methods are illustrated by two numerical examples.

Key words: Fuzzy integral equation, system of linear Fredholm integral equations of the second kind, successive approximation method, Taylor-successive approximation method.

INTRODUCTION

The fuzzy integral equation theory has been well developed (Friedman *et al.* 1999; Goetschel *et al.* 1986; Ma *et al.* 1999; Puri *et al.* 1986). Wu and Ma (Wu *et al.* 1990) investigated the fuzzy Fredholm integral equations for the first time. In recent years, some numerical methods have been introduced to solve linear fuzzy Fredholm integral equations of the second kind (FFIE-2). These methods can be found in (Abbasbandy *et al* 2006, 2007; Goghary *et al.* 2006; Ghanbari *et al*. 2009).

In the present work, we are going to construct two different numerical schemes for linear fuzzy Fredholm integral equations of the second kind based on successive approximation method. These schemes obtain the approximate-analytical solutions for FFIE-2.

The remainder of the paper is organized as follows. The next section presents the basic notations of fuzzy numbers, fuzzy functions and fuzzy integrals. Then linear fuzzy Fredholm integral equations of the second kind and its parametric form are discussed; and we observe the parametric form of the FFIE-2 is a system of linear Fredholm integral equations in the crisp case. Then we explain the successive and Taylor-successive approximation methods for the Fredholm integral equation of the second kind. Next we apply these methods on the system of Fredholm integral equations produced by FFIE-2, for some examples. Finally, we compare these results with the exact solutions.

2. Preliminaries:

Here we recall the basic notations for symmetric fuzzy numbers and symmetric fuzzy linear systems.

Definition 2.1:

Klir *et al.* 1997. A fuzzy number is a map $u: R \rightarrow I = [0,1]$ which satisfies:

i. u is upper semi-continuous;
ii. $u(x)=0$ outside some interval $[c,d] \subset R$;
iii. There exist real numbers a,b such that $c \le a \le b \le d$ where:
- $u(x)$ is monotonic increasing on $[c,a]$,
- $u(x)$ is monotonic decreasing on $[b,d]$,

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- $u(x)=1, a \leq x \leq b$.

The set of all the fuzzy numbers is denoted by E^1 . An equivalent parametric definition of fuzzy numbers is given in (Goetschel *et al.* 1986; Ma *et al.* 1999) as

Definition 2.2:

An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, that satisfies the following requirements

i $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over [0,1],

ii $\overline{u}(r)$ is a bounded left-continuous non-increasing function over [0,1],

iii $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

For arbitrary $u=(\underline{u}(r), \overline{u}(r))$, $v=(\underline{v}(r), \overline{v}(r))$ and $k \in R$ we define addition and multiplication by *k* as

$$\underline{(u+v)}(r) = (\underline{u}(r)+\underline{v}(r)) ,$$

$$\overline{(u+v)}(r) = (\overline{u}(r)+\overline{v}(r)) ,$$

$$\underline{ku}(r) = k(\underline{u}(r)) , \quad \overline{ku}(r) = k(\overline{u}(r)) \quad \text{if} \quad k \geq 0 ,$$

$$\overline{ku}(r) = k(\underline{u}(r)) , \quad \underline{ku}(r) = k(\overline{u}(r)) \quad \text{if} \quad k < 0 .$$

Definition 2.3:

For arbitrary fuzzy number $u, v \in E^1$, we use the distance (Goetschel *et al.* 1986)

$D(u,v) = sup_{0 \leq r \leq 1} \max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}$, and it is shown that (E^1, D) is a complete metric space (Puri *et al.* 1986).

Definition 2.4:

Friedman *et al.* 1999; Goetschel *et al.* 1986. Let $f:[a,b] \rightarrow E^1$, for each partition $P = t_0, \ldots, t_n$ of [*a,b*] and for arbitrary $\varepsilon_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$ suppose

$$R_p = \sum_{i=1}^{n} f(\varepsilon_i)(t_i - t_{i-1}) ,$$

$$\Delta := \max\{|t_i - t_{i-1}|\}, i = 1, \ldots, n$$

The definite integral of *f(t)* over [*a,b*] is

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_p ,$$

provided that the limit exists in the metric *D*. If the fuzzy function *f(t)* is continuous in the metric *D*, its definite integral exists (Goetschel *et al.* 1986), and also,

$$(\underline{\int_a^b f(t;r)dt}) = \int_a^b \underline{f}(t;r)dt ,$$

$\overline{(\int_a^b f(t;r)dt)} = \int_a^b \overline{f}(t;r)dt$.

3 Fuzzy Integral Equation:

The Fredholm integral equation of the second kind is (Hochstadt 1973)

$$F(t) = f(t) + \lambda \int_a^b K(s,t)F(s)ds \tag{3.1}$$

where $\lambda > 0, K(s,t)$ is an arbitrary given kernel function over the square $a \leq t, s \leq b$ and *f(t)* is given function of $t \in [a,b]$. If *f(t)* is a crisp function then the solution of (3.1) is crisp as well. However, if *f(t)* is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence equation of the second kind, where *f(t)* is a fuzzy function, are given in (Wu *et al.* 1990). Here, we introduce parametric form of a FFIE-2 with respect to Definition 2.4. Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{u}(t,r), \overline{u}(t,r)), 0 \leq r \leq 1$ and $t \in [a,b]$ are parametric form of *f(t)* and *u(t)*, respectively then, parametric form of FFIE-2 is as follows

$$\begin{aligned} \underline{u}(t,r) &= \underline{f}(t,r) + \lambda \int_a^b v_1(s,t,\underline{u}(s,r),\overline{u}(s,r))ds, \\ \overline{u}(t,r) &= \overline{f}(t,r) + \lambda \int_a^b v_2(s,t,\underline{u}(s,r),\overline{u}(s,r))ds. \end{aligned} \tag{3.2}$$

where

$$v_1(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} K(s,t)\underline{u}(s,r), & K(s,r) \geq 0, \\ K(s,t)\overline{u}(s,r), & K(s,r) < 0. \end{cases}$$

and

$$v_2(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} K(s,t)\overline{u}(s,r), & K(s,r) \geq 0, \\ K(s,t)\underline{u}(s,r), & K(s,r) < 0. \end{cases}$$

For each $0 \leq r \leq 1$ and $a \leq t \leq b$. We can see that (3.2) is a system of linear Fredholm integral equations in the crisp case for each $0 \leq r \leq 1$ and $a \leq t \leq b$. In next section, we explain two different successive schemes as numerical algorithms for approximating solution of this system of linear integral equations in the crisp case then, we find the approximate solutions for $\underline{u}(t,r)$ and $\overline{u}(t,r)$, for each $0 \leq r \leq 1$ and $a \leq t \leq b$.

4 Successive Schemes:

In this section, we first review the iterative method (Daftardar-Gejji *et al.* 2006) briefly. Let us consider the general functional equation

$$y = N(y) + f \tag{4.1}$$

where, *N* is a nonlinear operator and *f* is a known function. To find a solution, *y*, we assume that

$$y = \sum_{i=0}^{\infty} y_i \tag{4.2}$$

So, for the nonlinear operator N, we have

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right] \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we deduce that

$$\sum_{i=0}^{\infty} y_i = f + N(y_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right] \tag{4.4}$$

Regarding (4.4), we define

$$\begin{aligned} &y_0 = f, \\ &y_1 = N(y_0), \\ &y_2 = N(y_0 + y_1) - N(y_0), \\ &\vdots \\ &y_{m+1} = N(y_0 + \cdots + y_m) - N(y_0 + \cdots + y_{m-1}), \ m \geq 1. \end{aligned} \tag{4.5}$$

Hence,

$$y_0 + \cdots + y_{m+1} = N(y_0 + \cdots + y_m), \ m \geq 0 \tag{4.6}$$

and

$$y = f + \sum_{i=1}^{\infty} y_i \tag{4.7}$$

Now, we denote $M^{th}-$ order approximation of y by $Y_M = \sum_{i=0}^{M} y_i$, which y_i 's are obtained by (4.5). Therefore we have

$$\begin{aligned} &Y_0 = y_0 = f, \\ &Y_1 = y_0 + y_1 = f + N(y_0) = f + N(Y_0), \\ &Y_2 = y_0 + y_1 + y_2 = f + N(y_0 + y_1) = f + N(Y_1), \end{aligned} \tag{4.8}$$

and in the general form

$$\begin{aligned} Y_{M+1} &= y_0 + y_1 + \cdots + y_{M+1} = f + N(y_0) + N(y_0 + y_1) - N(y_0) \pm \cdots \\ &\quad + N(y_0 + y_1 + \cdots + y_M) - N(y_0 + y_1 + \cdots + y_{M-1}) \\ &= f + N(y_0 + y_1 + \cdots + y_M) = f + N(Y_M), \end{aligned} \tag{4.9}$$

Thus, the $M^{th}-$ order approximation of y, Y_M , is easily produced iteratively via the recurrence relation

$$Y_{M+1}(x) = f + N(Y_M(x)),\ M \geq 0 \tag{4.10}$$

with the initial value

$$Y_0(x) = f(x) \tag{4.11}$$

The method (4.10) is called the successive approximation scheme. Although the successive approximation scheme (4.10) has its advantage but it may be difficult to calculate the components $N(Y_M)$ and also require a large amount of computation. To overcome these disadvantages, the Taylor successive approximation scheme is proposed. Indeed, we use the Taylor series expansion in (4.10) and (4.11).

For a given function *g(x)*, we denote its $\nu^{th}-$ order Taylor series expansion at zero by $TL_\nu(g)$, *i.e.*,

$$TL_\nu(g(x)) = \sum_{k=0}^{\nu} \frac{g^{(k)}(0)}{k!} x^k \tag{4.12}$$

Substituting the $\nu^{th}-$ order Taylor series expansion of $Y_M(x)$ into (4.10), we derive

$$\begin{aligned} &Y_0(x) = f(x), \\ &Y_{M+1}(x) = f(x) + N(TL_\nu(Y_M(x))),\ M \geq 0 \end{aligned} \tag{4.13}$$

In fact, calculation of $N(Tl\boldsymbol{\nu}(Y_M))$ for $M = 0, 1, \ldots$ is simple, because $N(Tl\boldsymbol{\nu}(Y_M))$, for $M = 0,1,\ldots$ are now expressed as polynomials. Furthermore, $Tl\boldsymbol{\nu}(Y_M)$, for $M = 0,1,\ldots$ has at most *v+1* terms and the amount of computation is consequently reduced.

5 Main Results:

The following successive schemes can be obtain by substituting (4.10) and (4.13) for (3.1),

$$\begin{aligned} &\underline{u}_{M+1}(t,r) = \underline{f}(t,r) + \lambda \int_a^b v_1(s,t,\underline{u}_M(s,r),\overline{u}_M(s,r))ds, \\ &\underline{u}_0(t,r) = \underline{f}(t,r), \\ &\overline{u}_{M+1}(t,r) = \overline{f}(t,r) + \lambda \int_a^b v_2(s,t,\underline{u}_M(s,r),\overline{u}_M(s,r))ds, \\ &\overline{u}_0(t,r) = \overline{f}(t,r), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\underline{u}_{M+1}(t,r) = \underline{f}(t,r) + \lambda \int_a^b Tl_\nu\left(v_1(s,t,\underline{u}_M(s,r),\overline{u}_M(s,r))\right)ds, \\ &\underline{u}_0(t,r) = \underline{f}(t,r), \\ &\overline{u}_{M+1}(t,r) = \overline{f}(t,r) + \lambda \int_a^b Tl_\nu\left(v_2(s,t,\underline{u}_M(s,r),\overline{u}_M(s,r))\right)ds, \\ &\overline{u}_0(t,r) = \overline{f}(t,r). \end{aligned} \tag{5.2}$$

In the following theorem, we prove the convergence of these methods.

Theorem 5.1:

Consider the nonlinear functional equations (5.1) and (5.2). Define

$$N_1(\underline{u}(t,r)) = \lambda\int_a^b v_1(s,t,\underline{u}(s,r),\overline{u}(s,r))ds,\ \ N_2(\overline{u}(t,r)) = \lambda\int_a^b v_2(s,t,\underline{u}(s,r),\overline{u}(s,r))ds, \qquad (5.3)$$

And

$$NT_1(\underline{u}(t,r)) = \lambda\int_a^b T l_\nu\left(v_1(s,t,\underline{u}(s,r),\overline{u}(s,r))\right)ds\ ,\ NT_2(\overline{u}(t,r)) = \lambda\int_a^b T l_\nu\left(v_2(s,t,\underline{u}(s,r),\overline{u}(s,r))\right)ds. \qquad (5.4)$$

If N_1, N_2, NT_1 and NT_2 be contraction *i.e.*,

$$\| N_i(y) - N_i(u) \| \le k \| y-u \|,\ \ 0 < k < 1,\ \ i=1,2$$

and

$$\| NT_i(y) - NT_i(u) \| \le k_T \ \| y-u \|,\ \ 0 < k_T < 1,\ \ i=1,2$$

then the successive approximation methods (5.1) and (5.2) converge to the exact solution of equation (3.2).

Proof:

If we consider (5.3), then convergence of (5.1) is obtained from Theorem 4.1 in (Hosseini, 2010). In the same way, if we consider (5.4) then analogously to the demonstration of Theorem 4.1 in (Hosseini, 2010), we can prove the convergence of (5.2).

Example 5.1:

Consider the fuzzy Fredholm integral equation of the second kind (3.1) with

$$\underline{f}(t,r) = rt + 3/26 - 3/26\ r - 1/13\ t^2 - 1/13\ t^2 r \ ,$$

$$\overline{f}(t,r) = 2t - rt + 3/26\ r + 1/13\ t^2 r - 3/26 - 3/13\ t^2\ ,$$

and

$$K(s,t) = \frac{(s^2+t^2-2)}{13}\ ,\ 0 \le s\ , t \le 2\ ,$$

and $a=0, b=2$.

The exact solutions are given by

$$\underline{u}(t,r) = rt\ ,\ \overline{u}(t,r) = (2-r)t \text{ for all } r \in [0,1] \text{ and } t \in [0,2].$$

By applying the successive approximation scheme (5.1), one can see that, some first terms of the successive approximation series are as follows:

$$\underline{u}_0(t,r) = rt - (2078584526545145\ r)/18014398509481984 - (21651921307087\ rt^2)/281474976710656$$
$$-(21651921307087\ t^2)/281474976710656 + 2078584526545145/18014398509481984,$$

$$\underline{u}_1(t,r) = rt - (8527525974518937\ r)/28823037615171174 4 - (6822021010199451\ rt^2)/5764607523034234 88$$
$$- (2842508658172979\ t^2)/1441151880758558 72 + 2444557480616407/1441151880758558 72,$$

$\vdots$

$$\underline{u}_5(t,r) = rt - (198729780725357\ r)/4611686018427387904 - (3913448007831517\ rt^2)/590295810358705651712$$
$$- (3965828614918669\ t^2)/36893488147419103232 + 8253542583746889/590295810358705651712.$$

and

$$\overline{u}_0(t,r) = (2078584526545145\ r)/18014398509481984 + 2\ t - rt + (21651921307087\ rt^2)/281474976710656$$
$$- (4157168872946305\ t^2)/18014398509481984 - 2078584526545145/18014398509481984,$$

$$\overline{u}_1(t,r) = (8527525974518937\ r)/28823037615171174 4 + 2\ t - rt + (6822021010199451\ rt^2)/5764607523034234 88$$
$$- (1563379754789379\ t^2)/36028797018963968 - 3041484246951265/72057594037927936,$$

$\vdots$

$$\overline{u}_5(t,r) = (198729780725357\ r)/4611686018427387904 + 2\ t - rt + (3913448007831517\ rt^2)/590295810358705651712$$
$$- (1113752426110495\ t^2)/9223372036854775808 - 5327660160243063/73786976294838206464.$$

We use $\underline{u}_5(t,r)$ and $\overline{u}_5(t,r)$ as approximations for $\underline{u}(t,r)$ and $\overline{u}(t,r)$, respectively. The exact and obtained solutions of the fuzzy Fredholm integral equation at t = 1are compared in Table 1 and illustrated in Figure 1.

Table 1: The exact and approximate solutions for t = 1.

r	$\underline{u}_5(1,r)$	$\underline{u}(1,r)$	$\overline{u}_5(1,r)$	$\overline{u}(1,r)$
0	-9.3512 e-005	0.0	1.999807	2.0
0.1	0.09990152	0.1	1.899812	1.9
0.2	0.1998965	0.2	1.799817	1.8
0.3	0.2998916	0.3	1.699822	1.7
0.4	0.3998866	0.4	1.599827	1.6
0.5	0.4998816	0.5	1.499832	1.5
0.6	0.5998767	0.6	1.399837	1.4
0.7	0.6998717	0.7	1.299842	1.3
0.8	0.7998667	0.8	1.199847	1.2
0.9	0.8998617	0.9	1.099852	1.1
1	0.9998568	1.0	0.9998568	1.0

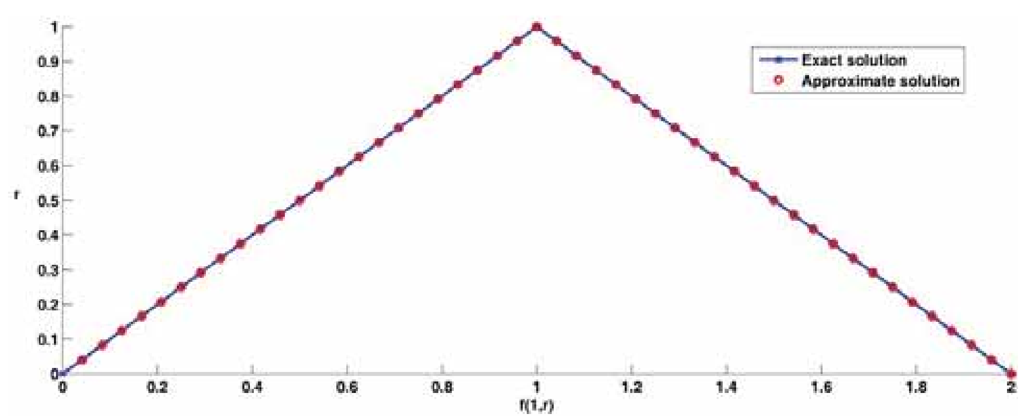

Fig. 1: The exact and approximate solutions for t = 1

Example 5.2: Consider the fuzzy Fredholm integral equation (3.1) with

$$\underline{f}(t,r) = sin(t/2)(13/15(r^2+r) + 2/15(4-r^3-r))$$

$$\overline{f}(t,r) = sin(t/2)(2/15(r^2+r) + 13/15(4-r^3-r))$$

and

$$K(s,t) = 0.1sin(s)sin(t/2), \quad 0 \le s,t \le 2\pi,$$

and $a = 0, b = 2\pi$

The exact solutions are given by

$\underline{u}(t,r) = (r^2+r)sin(t/2)$, $\overline{u}(t,r) = (4-r^3-r)sin(t/2)$, for all $r \in [0,1]$ and $t \in [0,2\pi]$.

In this example, we apply the Taylor-successive approximation scheme (5.2). Some first terms of the Taylor-successive approximation series are as follows:

$$\underline{u}_0(t,r) = sin(t/2)(0.86666667r^2 - 0.13333333r^3 + 0.73333333r + 0.53333333),$$

$$\underline{u}_1(t,r) = (sin(t/2)(34747059666704991\, r^2 - 1276381347322934\, r^3 + 33470677851007697\, r + 5105525893694896))/36028797018963968,$$

$$\vdots$$

$$\underline{u}_5(t,r) = (sin(t/2)(737756616492201482 33\, r^2 - 347362078363460\, r^3 + 73775313525365043200\, r + 1390186478368404))/73786976294838206464.$$

and

$$\overline{u}_0(t,r) = -sin(t/2)(0.86666667\, r^3 - 0.13333333\, r^2 + 0.73333333\, r - 3.4666667),$$

$$\overline{u}_1(t,r) = -(sin(t/2)(277976476180718423\, r^3 - 10211051355044222\, r^2 + 267765423672752708\, r - 1111905916540319204))/288230376151711744,$$

$$\vdots$$

$$\overline{u}_5(t,r) = -(sin(t/2)(73775661649220148233\, r^3 - 347362078363460\, r^2 + 73775313525365043200\, r - 295102648072472568676))/73786976294838206464.$$

We use $\underline{u}_5(t,r)$ and $\overline{u}_5(t,r)$ as approximations for $\underline{u}(t,r)$ and $\overline{u}(t,r)$, respectively. The exact and approximate solutions of the fuzzy Fredholm integral equation at $t = \pi$ are compared in Table 2 and illustrated in Figure 2.

6 Conclusions:

In this paper, we have outlined two successive schemes for solving fuzzy Fredholm integral equation of the second kind based on parametric form of fuzzy numbers. The results obtained by these schemes have been well adapted to the exact solutions. Moreover, the convergence theorem has been presented for these methods.

Table 2: The exact and approximate solutions for $t = \pi$

r	$\underline{u}_5(\pi, r)$	$\underline{u}(\pi, r)$	$\overline{u}_5(\pi, r)$	$\overline{u}(\pi, r)$
0	0.0000188405	0	3.9994	4
0.1	0.110001	0.11	9.8984	3.899
0.2	0.239981	0.24	3.79142	3.792
0.3	0.389957	0.39	3.67244	3.673
0.4	0.559931	0.56	6.53546	3.536
0.5	0.749901	0.75	3.37449	3.375
0.6	0.959868	0.96	3.18352	3.184
0.7	1.18983	1.19	2.95655	2.957
0.8	1.43979	1.44	2.68759	2.688
0.9	1.70975	1.71	2.37064	2.371
1	1.9997	2.0	1.9997	2

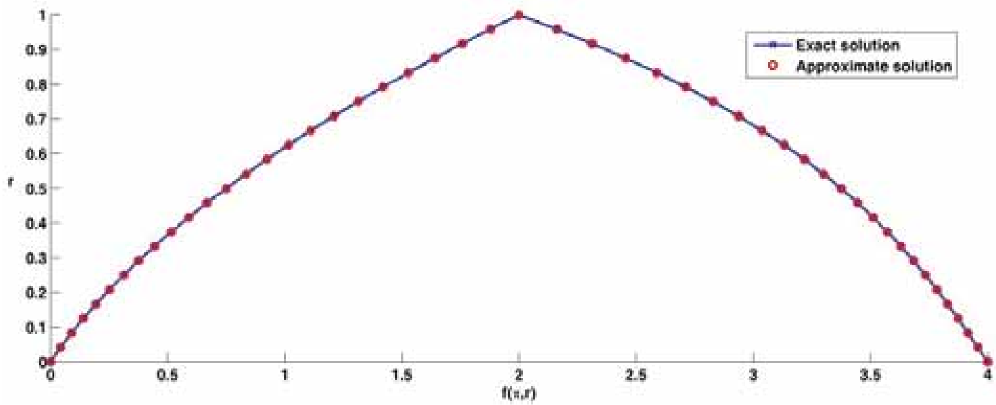

Fig. 2: The exact and approximate solutions for $t = \pi$

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