Neumann Asymptotic Eigenvalues of Sturm-liouville Problem with Three Turning Points

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Abstract: We consider the differential equation

\[-y'' + q(x)y = \lambda \phi^2(x)y, \quad x \in I = [0,1] \quad (*)\]

In this paper we study the asymptotic eigenvalues of Sturm-Liouville problem with Neumann conditions in three turning points case, that are here, the zeros of \( \phi(x) \) in (*). We find second term of asymptotic Neumann eigenvalues with three turning points case.

Key words: Turning point, Eigenvalues, Neumann conditions.

INTRODUCTION

We consider the boundary values problems which are the following form:

\[-y'' + q(x)y = \lambda \phi^2(x)y, \quad x \in I = [0,1] \quad (1)\]

where \( \lambda = \rho^2 \) is the spectral parameter, \( \phi^2 \) and \( q \) are real functions. We suppose that

\[\phi^2(x) = \prod_{i=1}^{3} (x - x_i)^{\ell_i} \phi_0(x) \quad (2)\]

where \( 0 < x_1 < x_2 < x_3 < 1 \), \( \ell_1, \ell_2, \ell_3 \in \mathbb{N} \), \( \phi_0(x) > 0 \) for \( x \in I = [0,1] \) and \( \phi_0 \) is twice continuously differentiable function on \( I \). On the other words, \( \phi_0 \) has in \( I \) there zeros \( x_i \) of order \( \ell_i \), \( i = 1, 2, 3 \) where \( \ell_1 \) is even, \( \ell_2 \) is odd and \( \ell_3 \) is even. Let \( \varepsilon > 0 \) be fixed sufficiently small and let

\[D_{i,\varepsilon} = [x_i + \varepsilon, x_{i+1} - \varepsilon], i = 1, 2, D_{3,\varepsilon} = [x_3 + \varepsilon, 1]\]

\[I_{i,\varepsilon} = [x_{i+1} + \varepsilon, x_i - \varepsilon]U[x_i + \varepsilon, x_{i+1} - \varepsilon]U[x_{i+1} + \varepsilon, x_i - \varepsilon]. \quad (4)\]
In (Eberhard, 1994) distinguished four different type of turning points: for \( 1 \leq \nu \leq m \).

\[
T_\nu = \begin{cases} 
I, & \text{if } l_\nu \text{ is even and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{x_\nu}, \\
II, & \text{if } l_\nu \text{ is even and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{x_\nu}, \\
III, & \text{if } l_\nu \text{ is odd and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{x_\nu}, \\
IV, & \text{if } l_\nu \text{ is odd and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{x_\nu}.
\end{cases}
\]  

We know from [2] that \( x_1 \) is of type I, \( x_2 \) is type IV and \( x_3 \) is of type II.

\[
\mu_i = \frac{1}{2 + l_i}, \quad \theta = \min \{\mu_1, \mu_2, \mu_3\}
\]

we use for convenience the abbreviations \( \theta = \min\{\mu_1,\mu_2,\mu_3\} \)

\[
[1] = 1 + O\left(\frac{1}{\rho^\theta}\right). \tag{6}
\]

We know from [2] that the sectors \( S_{-1} \) is in form of

\[
S_{-1} = \left\{ \rho \left| \text{arg } \rho \in \left[-\frac{\pi}{4}, 0\right] \right. \right\}
\]

3. The Fundamental Systems:

Now, let \( W(x, \lambda) \) be the solution of equation (1). The fundamental system of solutions (FSS) for equations (1), when \( T_\nu = IV \) can be represented in the form (see[2] page 219)

\[
W_{v1}^{IV}(x, \rho) = \left\{ \begin{array}{ll}
\phi(x) & \\
\frac{1}{2} e^{\int\rho \mu(1) dt} & (x, \rho) \in D_{v-1,\varepsilon} \times S_{-1}
\end{array} \right. \tag{7}
\]

\[
W_{v2}^{IV}(x, \rho) = \left\{ \begin{array}{ll}
\phi(x) & \\
\frac{1}{2} e^{\int\rho \mu(1) dt} & (x, \rho) \in D_{v-1,\varepsilon}
\end{array} \right. \tag{8}
\]

Since \( x_2 \) is of type IV, we have the following FSS for \( \rho \in S_{-1} \)

\[
W_{v1}^{II}(x, \rho) = \left\{ \begin{array}{ll}
\phi(x) & \\
\frac{1}{2} e^{\int\rho \mu(1) dt} & (x \in D_{-1,\varepsilon})
\end{array} \right. \tag{9}
\]

\[
W_{v2}^{II}(x, \rho) = \left\{ \begin{array}{ll}
\phi(x) & \\
\frac{1}{2} e^{\int\rho \mu(1) dt} & (x \in D_{-1,\varepsilon})
\end{array} \right. \tag{10}
\]
If $x$ be a turning point of type I, then the estimates for $W_{\nu,1}(x, \rho)$, $W_{\nu,2}(x, \rho)$ are obtained from the corresponding estimates for $W_{\nu,1}(x, \rho)$, $W_{\nu,2}(x, \rho)$ by substituting there in $\rho$ by $\rho$. In this paper the FSS of 

(1) for the sector $S_{\pm}=\left\{ \rho \right\}$, $\rho \in \left[ \tfrac{-\pi}{4}, 0 \right]$ are the following form

$$ W_{1,1}(x, \rho)=\begin{cases} 
\left| \phi(x) \right|^2 e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & 0 \leq x < x_1 \\
\left| \phi(x) \right|^2 \csc \mu \rho e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_1 < x < x_2 
\end{cases} $$

(11)

$$ W_{2,1}(x, \rho)=\begin{cases} 
\left| \phi(x) \right|^2 e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & 0 \leq x < x_1 \\
\left| \phi(x) \right|^2 \sin \mu \rho e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_1 < x < x_2, 
\end{cases} $$

(12)

Since $x_2$ is of type IV

$$ V_{1,2}(x, \rho)=\begin{cases} 
\left| \phi(x) \right|^2 e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_2 < x < x_3 \\
\left| \phi(x) \right|^2 \csc \mu \rho_2 e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} - \frac{\pi}{2} [1] e^{-i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} & x_2 < x < x_3 
\end{cases} $$

(13)

$$ V_{2,2}(x, \rho)=\begin{cases} 
\left| \phi(x) \right|^2 e^{\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_1 < x < x_2 \\
2\left| \phi(x) \right|^2 \sin \mu \rho_2 e^{i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_2 < x < x_3, 
\end{cases} $$

(14)

Since $x_3$ is of type II we also have the following FSS

$$ U_{1,2}(x, \rho)=\begin{cases} 
\left| \phi(x) \right|^2 e^{i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_3 < x < x_3 \\
\left| \phi(x) \right|^2 \csc \mu \rho e^{i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] + i \cos \mu \rho e^{-i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} & x_3 < x < x_3 
\end{cases} $$

(15)

$$ U_{2,3}(x, \rho)=\begin{cases} 
\left| \phi(x) \right|^2 \csc \mu \rho e^{i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] + i \cos \mu \rho e^{-i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} & x_2 < x < x_3 \\
\left| \phi(x) \right|^2 \sin \mu \rho e^{i\int_{\nu}^{\rho} \left| \phi(x) \right| \text{d} \rho} [1] & x_3 < x < 1. 
\end{cases} $$

(16)

The Wronskian of FSS satisfies in following form

$$ W(\rho)=W(W_{1,1}(x, \rho), W_{2,1}(x, \rho))=-2 \rho [1], $$

$$ W(V_{1,2}(x, \rho), V_{2,2}(x, \rho))=-2 \rho [1], $$

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4. Asymptotic Form of the Solution:

Let us consider the differential equation (1) with following conditions

\[ C(0, \lambda) = 1, \ C'(0, \lambda) = 0. \]

By applying the FSS \{W_{1,1}(x, \rho), W_{2,1}(x, \rho)\}, for \( x \in I_{1,\varepsilon} \) we have

\[ C(x, \rho) = c_1 W_{1,1}(x, \rho) + c_2 W_{2,1}(x, \rho) \]

by derivation from \( C(x, \rho) \) we can write

\[ C'(x, \rho) = c_1 W'_{1,1}(x, \rho) + c_2 W'_{2,1}(x, \rho) \quad \text{for} \ x \in I_{1,\varepsilon} \]

We infer by using Cramer’s rule leads to the following equation

\[ C(x, \rho) = \frac{1}{W(\rho)} (W'_{2,1}(0, \rho) W_{1,1}(x, \rho) - W'_{1,1}(0, \rho) W_{2,1}(x, \rho)) \]

where \( W(\rho) = -2\rho[1] \). By substituting (11-12) into account we derive

\[ C(x, \rho) = \begin{cases} 
\frac{1}{2} |\phi(x)| \frac{1}{2} |\phi(0)| e^{\frac{|\beta(1)|}{2}} E_1(x, \rho), & 0 \leq x < x_1 \\
\frac{1}{2} |\phi(x)| \frac{1}{2} |\phi(0)| \csc \mu_i e^{\frac{|\beta(1)|}{2}} E_2(x, \rho), & x_1 < x < x_2 
\end{cases} \]

where \( E_1(x, \rho) = [1] + \sum_{r=1}^{v(x)} e^{\alpha_r \beta(1)} [b_{kn}(x)] \), \( \alpha_2 = -1 \), \( \alpha_0 = -\alpha_1 = 1 \), \( \beta_{k_{1,0}}(x) \neq 0 \),

\( 0 < \delta \leq \beta_{k_1}(x) < \beta_{k_2}(x) < \ldots < \beta_{k_{v(x)}}(x) \leq 2\max\{R(1), R(1)\} \) where integer-valued functions \( v \) and \( b_{kn} \) are constant in every interval \( D_{j,\varepsilon}, j = 1, 2, 3 \) and

\[ R_1(x) = \int_0^x \sqrt{\max\{0, -\phi(1)\}} dt, \quad R_2(x) = \int_0^x \sqrt{\max\{0, -\phi(1)\}} dt. \]

Similarly in order to find the solution in \((x_1, x_2), (x_2, x_3), (x_3, 1)\) by using Cramer’s rule we have

\[ C(x, \rho) = \frac{1}{4} |\phi(x)| \frac{1}{2} |\phi(0)| \csc \mu_i \csc \frac{\mu_2}{2} e^{\frac{\mu_2}{2} \phi(1) + \phi(0) - \frac{i\pi}{4} E_3(x, \rho), \ x_2 < x < x_3. \]

We obtained the leading term of \( C(x, \rho) \) in \((x_3, 1)\) as follows too

\[ C(x, \rho) = \frac{1}{4} |\phi(x)| \frac{1}{2} |\phi(0)| \csc \mu_i \csc \frac{\mu_2}{2} \csc \frac{\mu_3}{2} e^{\frac{\mu_2}{2} \phi(1) dt - \frac{i\pi}{4} E_3(x, \rho), \ x_3 < x < 1. \]

5. Derivative of Solutions and Asymptotic Eigenvalues:

Let us consider boundary value problem \( L_1 = L_1(\phi(x), q(x), b) \) for equation (1) with boundary conditions

\[ y(0, \lambda) = 1, \quad y'(0, 1) = 0, \quad y'(0, b) = 0. \]
The boundary value problem $L_1$ for $b_0(x_3,1)$ has a countable set of positive eigenvalue. We consider

$$R_1(\rho) = \csc \pi \mu_1 \csc \frac{\pi \mu_2}{2} e^{\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha}, \quad T_1(\rho) = \frac{1}{2} R_1(\rho) e^{-i \frac{\pi}{4}}$$

$$R_2(\rho) = \sin \pi \mu_1 \sin \frac{\pi \mu_2}{2} e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha}, \quad T_2(\rho) = \frac{1}{2} R_1(\rho) e^{-i \frac{\pi}{4}} + 2 R_2(\rho) e^{-i \frac{\pi}{4}}.$$

Now for fixed $x \in (x_2, x_3)$ and using (15-16) we determine the connection coefficients $B_1(\rho), B_2(\rho)$

$$C(x, \rho) = B_1(\rho) \mu_{13} + B_2(\rho) \mu_{23} \Rightarrow C'(x, \rho) = B_1(\rho) \mu_{13} + B_2(\rho) \mu_{23}. \quad (20)$$

The derivation of $u_{1,3}(x, \rho)$ and $u_{2,3}(x, \rho)$ are in the following form

$$C'(x, \rho) = \frac{i}{4} |\phi(\alpha)|^2 \left[ B_1(\rho) \csc \pi \mu_3 \left[ e^{\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} - i \cos \pi \mu_3 e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} \right] \right][1]$$

$$C'(x, \rho) = -i \rho \sin \pi \mu_3 |\phi(\alpha)|^2 e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} \quad (21)$$

By substituting (21) in (20) we obtain $C'(x, \rho)$ in the following form

$$C'(x, \rho) = \frac{i}{4} |\phi(\alpha)|^2 \left[ B_1(\rho) \csc \pi \mu_3 \left[ e^{\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} - i \cos \pi \mu_3 e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} \right] \right] - B_2(\rho) \sin \pi \mu_3 |\phi(\alpha)|^2 e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} \quad (22)$$

Such that $B_1(\rho)$ and $B_2(\rho)$ are in the following form

$$B_1(\rho) = \frac{1}{4} |\phi(\alpha)|^2 \left[ R_1(\rho) e^{\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} - i \cos \pi \mu_3 \left( \frac{1}{2} R_1(\rho) e^{\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha + i \frac{\pi}{4}} + 2 R_2(\rho) e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} \right) \right][1] \quad (23)$$

$$B_2(\rho) = \frac{i}{4} |\phi(\alpha)|^2 \left[ R_1(\rho) e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha + i \frac{\pi}{4}} + 4 R_2(\rho) e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} \right][1] \quad (24)$$

We suppose

$$L_1 = \left\{ R_1(\rho) e^{\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} \right\} - R_2(\rho) \cos \pi \mu_3 e^{-i \rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} - 2 i \cos \pi \mu_3 R_2(\rho) e^{-i \rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} \quad (25)$$

$$L_1 = \left\{ R_1(\rho) e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha + i \frac{\pi}{4}} + 4 R_2(\rho) e^{-\rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha - i \frac{\pi}{4}} \right\}[1] \quad (26)$$

$$H_1 = \csc \pi \mu_3 \left\{ e^{i \rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} - i \cos \pi \mu_3 e^{-i \rho \int_{\rho}^{1} |\phi(\alpha)| d\alpha} \right\}[1] \quad (27)$$
Thus, by using (25-28) and substituting them in (22) we obtain the leading term of \( C'(x, p) \) as follows

\[
C'(x, \rho) = \frac{i\rho}{4} \left[ \phi(0)^{1/2} \phi'(x)^{1/2} (H_1L_1 - H_2L_2) \right].
\]  

(29)

Now we account \( H_1L_1 \) and \( H_2L_2 \) in following form

\[
H_1L_1 = \csc \pi \mu_1 \{ R_1(\rho) e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - iR_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} + \frac{i}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - \frac{1}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} \}.
\]  

(30)

\[
H_2L_2 = \left( R_1(\rho) \sin \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - iR_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} + \frac{i}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - \frac{1}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} \right).
\]  

(31)

When \( \rho \to \infty \) then \( e^{-i\rho \int_{0}^{\pi/2} \phi(t) dt} \to 0 \) so, \( R_1(\rho) \to 0 \).

By minus (31) from (30) we have

\[
H_1L_1 - H_2L_2 = \csc \pi \mu_1 \{ R_1(\rho) e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - iR_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} + \frac{i}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - \frac{1}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} \}.
\]  

(32)

By substituting (32) in (29) leads to the equation

\[
C'(x, \rho) = \frac{i\rho}{4} \left[ \phi(0)^{1/2} \phi'(x)^{1/2} \csc \pi \mu_1 \{ R_1(\rho) e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - iR_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} + \frac{i}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - \frac{1}{2} R_1(\rho) \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} \} \right].
\]  

(33)

Consequently, we have \( C'(x, \rho) \) in the following form

\[
C'(x, \rho) = \frac{i\rho}{4} \left[ \phi(0)^{1/2} \phi'(x)^{1/2} \csc \pi \mu_1 \csc \frac{\pi \mu_1}{2} \csc \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} - \frac{1}{2} \cos \pi \mu_1 e^{i\rho \int_{0}^{\pi/2} \phi(t) dt} \right] \times E(x, \rho), \ x_1 < x < 1.
\]  

(34)
So that \(E(x, \rho)\) are defined as following form

\[
E(x, \rho) = [1 - \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + \frac{i\pi}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} - \frac{1}{2} \cos^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} - i \sin^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt}].
\] (36)

By applying \(C'(x, \rho) = 0\), consequently \(E(x, \rho) = 0\), therefore

\[
[1 - \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} - \frac{i\pi}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + \frac{1}{2} \cos^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + i \sin^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt}].
\] (37)

By using Moaver’s rule we have \(i = e^{\frac{i\pi}{2}}\), so we can rewrite

\[
[1 + \frac{1}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} - \frac{1}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + \frac{1}{2} \cos^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + i \sin^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt}]
\] (38)

We know \([1] = 1 + O\left(\frac{1}{\rho^2}\right)\) hence we have

\[
1 + O\left(\frac{1}{\rho^2}\right) + \frac{1}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} - \frac{1}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + \frac{1}{2} \cos^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + i \sin^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt}
\] (39)

\[
= e^{-2i\rho \int_0^\pi |\phi(t)| dt + \frac{i\pi}{2}} (\cos \pi \mu_3 + \sin^2 \pi \mu_3),
\] (40)

Such that \(K\) is a fixed number and are given in the following form

\[
K = 1 + \frac{1}{2} \cos \pi \mu_3 e^{-2i\rho \int_0^\pi |\phi(t)| dt} - \frac{1}{2} \cos \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + \frac{1}{2} \cos^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt} + i \sin^2 \mu_i e^{-2i\rho \int_0^\pi |\phi(t)| dt}
\] (41)

\[
\frac{K}{\cos \pi \mu_3 + \sin^2 \pi \mu_3} + O\left(\frac{1}{\rho^2}\right) = e^{-2i\rho \int_0^\pi |\phi(t)| dt + \frac{i\pi}{2}}
\] (42)

\[
L + O\left(\frac{1}{\rho^2}\right) = e^{-2i\rho \int_0^\pi |\phi(t)| dt + \frac{i\pi}{2} + 2k\pi}
\]
Here we called \( L = \frac{K}{\cos \pi \mu + \sin^2 \pi \mu} \)

\[
O\left(\frac{1}{\rho^{\theta}}\right) = -2i\rho \int_{x_3}^{x} \left| \phi(t) \right| dt + \frac{i\pi}{2} + 2k\pi i
\]  

(43)

by factoring \(-2i\), (43) can be written as follows

\[
O\left(\frac{1}{\rho^{\theta}}\right) = \rho \int_{x_3}^{x} \left| \phi(t) \right| dt - \frac{\pi}{4} - k\pi
\]  

(44)

where

\[
\rho = \frac{k\pi + \frac{\pi}{4}}{\int_{x_3}^{x} \left| \phi(t) \right| dt} + O\left(\frac{1}{k^{\theta}}\right),
\]  

(45)

since \( \rho = O(k) \), then the eigenvalues of equation (1) are obtained in the following form

\[
\rho_k = \frac{k\pi + \frac{\pi}{4}}{\int_{x_3}^{x} \left| \phi(t) \right| dt} + O\left(\frac{1}{k^{\theta}}\right)
\]  

(46)

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