Australian Journal of Basic and Applied Sciences, 5(5): 1189-1196, 2011
ISSN 1991-8178

Neumann Asymptotic Eigenvalues of Sturm-liouville Problem with Three Turning Points

E.A.Sazgar

Department of Mathematics, Yerevan state University, Armenia.

Abstract: We consider the differential equation

$-y''+q(x)y=\lambda\phi^2(x)y, \qquad x\in I=[0,1] \quad (*).$

In this paper we study the asymptotic eigenvalues of Sturm-Liouville problem with Neumann conditions in three turning points case, that are here, the zeros of φ(x) in (*). We find second term of asymptotic Neumann eigenvalues with three turning points case.

Key words: Turning point, Eigen values, Neumann conditions.

INTRODUCTION

We consider the boundary values problems which are the following form:

$$-y''+q(x)y=\lambda\phi^2(x)y, \quad x\in I=[0,1] \tag{1}$$

where $\lambda = \rho^2$ is the spectral parameter; φ^2 and q are real functions. We suppose that

$$\phi^2(x)=\prod_{i=1}^{3}(x-x_i)^{\ell_i}\,\phi_0(x) \tag{2}$$

where $0 < x_1 < x_2 < x_3 < 1$, L_i ε N, $\varphi_0(x) > 0$ for $x \in I = [0, 1]$ and φ_0 is twice continuously differentiable function on I. On the other words, φ_2 has in I there zeros x_i, $i = 1, 2, 3$ of order ℓ_i, the zeros x_i of φ_2 are called turning points. Differential equations with turning points play an important role in mathematics and other branches of natural sciences for example in elasticty, optics, geophysics. In fact, differential equations with turning points have various applications. The importance of asymptotic analysis in obtaining the solution of Sturm-Liouville equation with turning points was realized by Leuny, Olver, Eberhard, Freiling in (1997). Neamaty and Sazgar (2009) authors, obtained asymptotic Neumann eigenvalues of Sturm-Liouville problem in two turning points. In this paper we obtained the asymptotic eigenvalues of equation (1) in tree turning points case with Neumann condition.

2. Notations:

In the real second-order differential equations

$$-y''+q(x)y=\lambda\phi^2(x)y \tag{3}$$

φ^2 has in I, there zeros x_i of order l_i, i = 1, 2, 3 where ℓ_1 is even, ℓ_2 is odd and ℓ_3 is even. Let ε > 0 be fixed sufficiently small and let

$$D_{i,\varepsilon}=[x_i+\varepsilon,x_{i+1}-\varepsilon], i=1,2 \;\; D_{3,\varepsilon}=[x_3+\varepsilon,1]$$

$$I_{i,\varepsilon}=[x_{i-1}+\varepsilon,x_i-\varepsilon]U[x_i+\varepsilon,x_{i+\varepsilon}]U[x_{i+\varepsilon},x_{i+1}-\varepsilon]. \tag{4}$$

Corresponding Author: E.A.Sazgar, Department of Mathematics, Yerevan state University, Armenia.
E-mail: EA. Sazgar@Gmail.com

In (Eberhard, 1994) distinguished four different type of turning points: for $1 \leq \nu \leq m$.

$$T_v = \begin{cases} I, if\ l_v\ is\ even\ and\ \phi^2(x)(x-x_v)^{-l_v} < 0\ in\ I_{v\varepsilon}, \\ II, if\ l_v\ is\ even\ and\ \phi^2(x)(x-x_v)^{-l_v} < 0\ in\ I_{v\varepsilon}, \\ III, if\ l_v\ is\ odd\ and\ \phi^2(x)(x-x_v)^{-l_v} < 0\ in\ I_{v\varepsilon}, \\ IV, if\ l_v\ is\ odd\ and\ \phi^2(x)(x-x_v)^{-l_v} < 0\ in\ I_{v\varepsilon}. \end{cases} \tag{5}$$

We know from [2] that x_1 is of type I, x_2 is type IV and x_3 is of type II.

$$\mu_i = \frac{1}{2+l_i},\ \theta = \min\{\mu_1, \mu_2, \mu_3\}$$

we use for convenience the abbreviations $\theta = \min\{\mu_1,\mu_2,\mu_3\}$

$$[1] = 1 + O\left(\frac{1}{\rho^{\theta}}\right). \tag{6}$$

We know from [2] that the sectors S_{-1} is in form of

$$S_{-1} = \left\{\rho \middle| \arg\rho \in |-\frac{\pi}{4}, 0]\right\}$$

3. The Fundamental Systems:

Now, let W (x,λ) be the solution of equation (1). The fundamental system of solutions (FSS) for equations (1), when Tν = IV can be represented in the form (see[2] page 219)

$$W_{v1}^{IV}(x,\rho) = \{|\phi(x)|^{-\frac{1}{2}} e^{\rho\int_{xv}^{x}|\phi(t)|dt} [1] \quad (x,p) \in D_{v-1,\varepsilon} \times s_{-1} \tag{7}$$

$$W_{v2}^{IV}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1]\ x \in D_{v-1,\varepsilon} \\ |\phi(x)|^{-\frac{1}{2}} \times 2\sin\frac{\pi\mu_v}{2} e^{-ip\int_{xv}^{x}|\phi(t)|dt - \frac{\pi}{4}}[1] x \in D_{v,\varepsilon} \end{cases} \tag{8}$$

Since x_2 is of type IV, we have the following FSS for $\rho \in S_{-1}$

$$W_{v1}^{II}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1] \quad x \in D_{v-1,\varepsilon} \\ |\phi(x)|^{-\frac{1}{2}} \csc\pi\mu_v e^{i\rho\int_{xv}^{x}|\phi(t)|dt}[1] + i\cos\pi\mu_v e^{i\rho\int_{xv}^{x}|\phi(t)|dt}[1] \quad x \in D_{v,\varepsilon} \end{cases} \tag{9}$$

$$W_{v2}^{II}(x,\rho) = \begin{cases} |\phi(x)|^{-\frac{1}{2}} \csc\pi\mu_v e^{i\rho\int_{xv}^{x}|\phi(t)|dt}[1] + i\cos\pi\mu_v e^{i\rho\int_{xv}^{x}|\phi(t)|dt}[1] \quad x \in D_{v-1,\varepsilon} \\ |\phi(x)|^{\frac{1}{2}} (\sin\pi\mu_v e^{i\rho\int_{xv}^{x}|\phi(t)|dt}[1]) \quad x \in D_{v,\varepsilon} \end{cases} \tag{10}$$

If x_v be a turning point of type I, then the estimates for $W^t_{v1}(x,\rho)$, $W^t_{v2}(x,\rho)$ are obtained from the corresponding estimates for $W^t_{v1}(x,\rho)$, $W^t_{v2}(x,\rho)$ by substituting there in ρ by iρ. In this paper the FSS of (1) for the sector $S_{-1}=\left\{\rho \mid \arg\rho\in\left|-\frac{\pi}{4},0\right]\right\}$ are the following form

$$W_{1,1}(x,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1] & 0\le x<x_1\\ |\phi(x)|^{-\frac{1}{2}}\csc\pi\mu_1 e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1] & x_1<x<x_2\end{cases} \tag{11}$$

$$W_{2,1}(x,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1] & 0\le x<x_1\\ |\phi(x)|^{-\frac{1}{2}}\sin\pi\mu_1 e^{-\rho\int_{xv}^{x}|\phi(t)|dt}[1] & x_1<x<x_2,\end{cases} \tag{12}$$

Since x_2 is type of IV

$$V_{1,2}(x,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1] & x_2<x<x_3\\ |\phi(x)|^{-\frac{1}{2}}\csc\frac{\pi\mu_2}{2}e^{i\rho\int_{xv}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1]+e^{-i\rho\int_{xv}^{x}|\phi(t)|dt} & x_2<x<x_3\end{cases} \tag{13}$$

$$V_{2,2}(x,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{\rho\int_{xv}^{x}|\phi(t)|dt}[1] & x_1\le x<x_2\\ 2|\phi(x)|^{-\frac{1}{2}}\sin\frac{\pi\mu_2}{2}e^{i\rho\int_{xv}^{x}|\phi(t)|dt}[1] & x_2<x<x_3,\end{cases} \tag{14}$$

Since x_3 is of type of II we also have the following FSS

$$U_{1,3}(x,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}e^{i\rho\int_{x3}^{x}|\phi(t)|dt}[1] & x_2<x<x_3\\ |\phi(x)|^{-\frac{1}{2}}\csc\pi\mu_3 e^{i\rho\int_{x3}^{x}|\phi(t)|dt}[1]+i\cos\pi\mu_3 e^{-i\rho\int_{x3}^{x}|\phi(t)|dt}[1] & x_3<x<1\end{cases} \tag{15}$$

$$U_{2,3}(x,\rho)=\begin{cases}|\phi(x)|^{-\frac{1}{2}}\csc\pi\mu_3 e^{i\rho\int_{x3}^{x}|\phi(t)|dt}[1]+i\cos\pi\mu_3 e^{-i\rho\int_{x3}^{x}|\phi(t)|dt}[1] & x_2<x<x_3\\ |\phi(x)|^{-\frac{1}{2}}\sin e^{i\rho\int_{x3}^{x}|\phi(t)|dt}[1] & x_3<x<1.\end{cases} \tag{16}$$

The Wronskian of FSS satisfies in following form

$$W(\rho)=W(W_{1,1}(x,\rho),W_{2,1}(x,\rho))=-2\rho[1],$$

$$W(V_{1,2}(x,\rho),V_{2,2}(x,\rho))=-2\rho[1],$$

$$W(U_{1,3}(x,\rho),U_{2,3}(x,\rho)=2i\rho[1], as\ \rho\to\infty. \quad (17)$$

4. Asymptotic Form of the Solution:

Let us consider the differential equation (1) with following conditions

$$C(0,\lambda)=1,\ C'(0,\lambda)=0.$$

By applying the FSS {$W_{1,1}(x, \rho)$, $W_{2,1}(x, \rho)$}, for $x \in I_{1,\varepsilon}$ we have

$$C(x,\rho)=c_1 W_{1,1}(x,\rho)+c_2 W_{2,1}(x,\rho)$$

by derivation from C (x, ρ) we can write

$$C'(x,\rho)=c_1 W'_{1,1}(x,\rho)+c_2 W'_{2,1}(x,\rho)\ \ for\ x\in I_{1,\in}$$

We infer by using Cramer's rule leads to the following equation

$$C(x,\rho)=\frac{1}{W(\rho)}(W'_{2,1}(0,\rho)W_{1,1}(x,\rho)-W'_{1,1}(0,\rho)W_{2,1}(x,\rho)), \quad (18)$$

where W(ρ) = -2ρ[1]. By substituting (11-12) in to account we derive

$$C(x,\rho)=\begin{cases}\frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}e^{\rho\int_0^x|\phi(t)|dt}E_k(x,\rho)\ 0\le x<x_1\\ \frac{1}{2}|\phi(x)|^{-\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1 e^{\rho\int_0^x|\phi(t)|dt}E_k(x,\rho)\ x_1<x<x_2\end{cases} \quad (19)$$

where E_k (x, ρ) = [1] + $+\sum_{n=1}^{v(x)} e^{\rho\alpha_k\beta_{kn}^{(x)}}[b_{kn}(x)]$, $\alpha_2 = \beta_1 = -1$, $\alpha_0 = -\alpha_{-1} = I$, $\beta_{kv(x)}(x) \neq 0$,

$0 < \delta \le \beta_{k1}(x) < \beta_{k2}(x) < ... < \beta_{kv(x)}(x) \le 2\max\{R+(1), R_-(1)\}$ where integer-valued functions ν and b_{kn} are constant in every interval $D_{j,\varepsilon}$, j = 1, 2, 3 and

$$R_+(x)=\int_0^x\sqrt{\max\{0,\phi^2(t)\}}dt,\ R_-(x)=\int_0^x\sqrt{\max\{0,-\phi^2(t)\}}dt.$$

Similarly in order to find the solution in (x_1, x_2), (x_2, x_3), (x_3, 1) by using Cramer's rule we have

$$C(x,\rho)=\frac{1}{4}|\phi(x)|^{-\frac{1}{2}}|\phi(x)|^{\frac{1}{2}}\csc\pi\mu_1\csc\frac{\pi\mu_2}{2}e^{p\int_0^{x_2}|\phi(t)|dt+i\rho\int_{x_2}^{x}|\phi(t)|dt-\frac{i\pi}{4}}E_k(x,\rho),\ x_2<x<x_3.$$

We obtained the leading term of C(x, ρ) in (x_3, 1) as follows too

$$C(x,\rho)=\frac{1}{4}|\phi(x)|^{-\frac{1}{2}}|\phi(x)|^{\frac{1}{2}}\csc\pi\mu_1\csc\frac{\pi\mu_2}{2}\csc\pi\mu_3\, e^{\rho\int_{x_2}^{x}|\phi(t)|dt-\frac{i\pi}{4}}E_k(x,\rho), x_3<x<1$$

5. Derivative of Solutions and Asymptotic Eigenvalues:

Let us consider boundary value problem $L_1 = L_1$ (φ^2(x), q (x), b) for equation (1) with boundary conditions

y (0, λ) = 1,   y\` (0, 1) = 0,     y\` (0, b) = 0.

The boundary value problem L_1 for $b \in (x_3, 1)$ has a countable set of positive eigenvalue. We consider

$$R_1(\rho)=\csc\pi\mu_1 \csc\frac{\pi\mu_2}{2} e^{\rho\int_0^{x2}|\phi(t)|dt},\ T_1(\rho)=\frac{1}{2}R_1(\rho)e^{-i\frac{\pi}{4}}$$

$$R_2(\rho)=\sin\pi\mu_1 \sin\frac{\pi\mu_2}{2} e^{-\rho\int_0^{x2}|\phi(t)|dt},\ T_2(\rho)=\frac{1}{2}R_1(\rho)e^{-i\frac{\pi}{4}}+2R_2(\rho)e^{-i\frac{\pi}{4}}.$$

Now for fixed $x \in (x_2, x_3)$ and using (15-16) we determine the connection coefficients $B_1(\rho)$, $B_2(\rho)$

$$C(x,\rho)=B_1(\rho)\mu_{13}+B_2(\rho)\mu_{23} \Rightarrow C'(x,\rho)=B_1(\rho)\mu'_{13}+B_2(\rho)\mu'_{23}. \tag{20}$$

The derivation of $u_{1,3}(x, \rho)$, and $u_{2,3}(x, \rho)$ are in the following form

$$\begin{cases} \mu'_{13}(x,\rho)=i\rho|\phi(x)|^{\frac{1}{2}}\csc\pi\mu_3\{e^{\rho\int_{x3}^{x}|\phi(t)|dt}-i\cos\pi\mu_3 e^{-\rho\int_{x3}^{x}|\phi(t)|dt}\}[1] \\ \mu'_{13}(x,\rho)=-i\rho\sin\pi\mu_3|\phi(x)|^{\frac{1}{2}}e^{-\rho\int_{x3}^{x}|\phi(t)|dt}[1]\ x_3<x<1. \end{cases} \tag{21}$$

By substituting (21) in (20) we obtain $C'(x, \rho)$ in the following form

$$C'(x,\rho)=i\rho|\phi_{(x)}|^{\frac{1}{2}}\left\{B_1(\rho)\csc\pi\mu_3\left[e^{\rho\int_{x3}^{x}|\phi(t)|dt}-i\cos\pi\mu_3 e^{-\rho\int_{x3}^{x}|\phi(t)|dt}\right][1]-B_2(\rho)\sin\pi\mu_3|\phi_{(x)}|^{\frac{1}{2}}e^{-\rho\int_{x3}^{x}|\phi(t)|dt}[1]\right\}, \tag{22}$$

Such that $B_1(\rho)$ and $B_2(\rho)$ are in the following form

$$B_1(\rho)=\frac{1}{4}|\phi(x)|^{\frac{1}{2}}\left\{R_1(\rho)e^{\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}}-i\cos\pi\mu_3\left(\frac{1}{2}R_1(\rho)e^{\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}}+2R_2(\rho)e^{-\rho\int_{x2}^{x3}|\phi(t)|dt-\frac{i\pi}{4}}\right)\right\}[1] \tag{23}$$

$$B_2(\rho)=\frac{i}{4}|\phi(x)|^{\frac{1}{2}}\left\{R_1(\rho)e^{-i\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}}+4R_2(\rho)e^{-\rho\int_{x2}^{x3}|\phi(t)|dt-\frac{i\pi}{4}}\right\}[1]. \tag{24}$$

We suppose

$$L_1=\left\{R_1(\rho)e^{\rho\int_{x2}^{x3}|\phi(t)|dt-\frac{i\pi}{4}}-\frac{i}{2}R_1(\rho)\cos\pi\mu_3 e^{-i\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}}-2i\cos\pi\mu_3 R_2(\rho)e^{-i\rho\int_{x2}^{x3}|\phi(t)|dt-\frac{i\pi}{4}}\right\} \tag{25}$$

$$L_1=\left(R_1(\rho)e^{i\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}}+4R_2(\rho)e^{-\rho\int_{x2}^{x3}|\phi(t)|dt-\frac{i\pi}{4}}\right)[1], \tag{26}$$

$$H_1=\csc\pi\mu_3\left(e^{i\rho\int_{x2}^{x3}|\phi(t)|dt}-i\cos\pi\mu_3 e^{-i\rho\int_{x2}^{x3}|\phi(t)|dt}\right)[1], \tag{27}$$

$$H_2 = \sin \pi\mu_3 e^{-i\rho\int_{x2}^{x3}|\phi(t)|dt} [1]. \tag{28}$$

Thus, by using (25-28) and substituting them in (22) we obtain the leading term of C' (x, p) as follows

$$C'(x,\rho)=\frac{i\rho}{4}|\phi(0)|^{\frac{1}{2}}|\phi(x)|^{\frac{1}{2}}(H_1L_1 - H_2L_2). \tag{29}$$

Now we account H_1L_1 and $H_2 L_2$ in following form

$$\begin{aligned} H_1L_1 = \csc\pi\mu_3 \{ & R_1(\rho)e^{i\rho\int_{x2}^{x}|\phi(t)|dt-\frac{i\pi}{4}} - iR_1(\rho)\cos\pi\mu_3 e^{i\rho\int_{x2}^{x3}|\phi(t)|dt - i\rho\int_{x2}^{x}|\phi(t)|dt-\frac{i\pi}{4}} - \frac{i}{2}R_1(\rho)\cos\pi\mu_3 e^{i\rho\int_{x3}^{x}|\phi(t)|dt - i\rho\int_{x2}^{x^3}|\phi(t)|dt-\frac{i\pi}{4}} \\ & -\frac{1}{2}R_1(\rho)\cos^2\pi\mu_3 e^{-i\rho\int_{x2}^{x}|\phi(t)|dt - i\rho\int_{x2}^{x^3}|\phi(t)|dt+\frac{i\pi}{4}} - 2iR_2(\rho)\cos\pi\mu_3 e^{i\rho\int_{x3}^{x}|\phi(t)|dt-\frac{i\pi}{4}} - 2iR_2(\rho)\cos^2\pi\mu_3 e^{-i\rho\int_{x3}^{x}|\phi(t)|dt-\frac{i\pi}{4}} \}[1] \end{aligned} \tag{30}$$

$$H_2L_2 = \left(R_1(\rho)\sin\pi\mu_3 e^{-i\rho\int_{x3}^{x}|\phi(t)|dt + \rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}} + 4R_2(\rho)\sin\pi\mu_3 e^{-i\rho\int_{x3}^{x}|\phi(t)|dt-\frac{i\pi}{4}} \right)[1]. \tag{31}$$

When $\rho \to \infty$ then $e^{-\rho\int_0^x|\phi(t)|dt} \to 0 \, so, R_2(\rho) \to 0.$

By minus (31) from (30) we have

$$\begin{aligned} H_1L_1 - H_2L_2 \csc\pi\mu_3 R_1(\rho)\{ & e^{i\rho\int_{x2}^{x}|\phi(t)|dt-\frac{i\pi}{4}} - i\cos\pi\mu_3 e^{i\rho\int_{x2}^{x3}|\phi(t)|dt - i\rho\int_{x3}^{x}|\phi(t)|dt-\frac{i\pi}{4}} - \frac{i}{2}\cos\pi\mu_3 \\ & e^{i\rho\int_{x3}^{x}|\phi(t)|dt - i\rho\int_{x2}^{x^3}|\phi(t)|dt+\frac{i\pi}{4}} - \frac{1}{2}\cos^2\pi\mu_3 e^{i\rho\int_{x2}^{x}|\phi(t)|dt+\frac{i\pi}{4}} - \sin\pi\mu_3 e^{i\rho\int_{x3}^{x}|\phi(t)|dt + i\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}} \}[1] \end{aligned} \tag{32}$$

By substituting (32) in (29) leads to the equation

$$\begin{aligned} C'(x,\rho)=\frac{i\rho}{4}|\phi(0)|^{\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_3 R_1(\rho)\{ & e^{i\rho\int_{x2}^{x}|\phi(t)|dt-\frac{i\pi}{4}} - i\cos\pi\mu_3 e^{i\rho\int_{x2}^{x3}|\phi(t)|dt + i\rho\int_{x}^{x3}|\phi(t)|dt-\frac{i\pi}{4}} - \frac{i}{2}\cos\pi\mu_3 \\ & e^{i\rho\int_{x3}^{x2}|\phi(t)|dt + i\rho\int_{x2}^{x}|\phi(t)|dt - i\rho\int_{x2}^{x}|\phi(t)|dt - i\rho\int_{x}^{x3}|\phi(t)|dt+\frac{i\pi}{4}} \\ & -\frac{1}{2}\cos^2\pi\mu_3 e^{i\rho\int_{x2}^{x}|\phi(t)|dt+\frac{i\pi}{4}} - \sin^2\pi\mu_3 e^{-i\rho\int_{x3}^{x}|\phi(t)|dt + i\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{4}} \} \end{aligned} \tag{33}$$

$$\begin{aligned} C(x,\rho)=\frac{i\rho}{4}|\phi(0)|^{\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1\csc\pi\mu_3 e^{\rho\int_0^{x2}|\phi(t)|dt + i\rho\int_{x2}^{x}|\phi(t)|dt-\frac{i\pi}{4}} \times \{ & [1] - i\cos\pi\mu_3 e^{-2i\rho\int_{x3}^{x}|\phi(t)|dt} - \frac{i}{2}\cos\pi\mu_3 e^{-2i\rho\int_{x2}^{x3}|\phi(t)|dt+\frac{i\pi}{2}} - \\ & \frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x2}^{x}|\phi(t)|dt+\frac{i\pi}{4}} - \sin^2\pi\mu_3 e^{-2i\rho\int_{x3}^{x}|\phi(t)|dt+\frac{i\pi}{2}} \} \end{aligned} \tag{34}$$

Consequently, we have C' (x, ρ) in the following form

$$C'(x,\rho)=\frac{i\rho}{4}|\phi(0)|^{\frac{1}{2}}|\phi(0)|^{\frac{1}{2}}\csc\pi\mu_1 \csc\frac{\pi\mu_2}{2}\csc\pi\mu_3 e^{\rho\int_0^{x2}|\phi(t)|dt + i\rho\int_{x2}^{x}|\phi(t)|dt-\frac{i\pi}{4}} \times E(x,\rho), \quad x_3 < x < 1, \tag{35}$$

So that E(x, ρ) are defined as following form

$$E(x,\rho)=[1]-i\cos\pi\mu_3\, e^{-2i\rho\int_{x_2}^{x}|\phi(t)|dt}+\frac{i\pi}{2}i\cos\pi\mu_3 e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt}-\frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x_2}^{x}|\phi(t)|dt}-i\sin^2\pi\mu_3 e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt}. \quad (36)$$

By applying C'*(x, ρ)* = 0, consequently E *(x, ρ)* = 0, therefore

$$[1]-i\cos\pi\mu_3\, e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt}-\frac{i\pi}{2}i\cos\pi\mu_3 e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt}+\frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x_2}^{x}|\phi(t)|dt}+i\sin^2\pi\mu_3 e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt}. \quad (37)$$

By using Moaver's rule we have $i=e^{\frac{i\pi}{2}}$, so we can rewrite

$$\begin{aligned}[1]+\frac{1}{2}\cos\pi\mu_3\, e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt} &= i\cos\pi\mu_3 e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt}+\frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt} \\ &= \cos\pi\mu_3\, e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt+\frac{i\pi}{4}}+\frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt+\frac{i\pi}{4}}+i\sin^2\pi\mu_3 e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt+\frac{i\pi}{4}}.\end{aligned} \quad (38)$$

We know [1] = $1+O(\frac{1}{\rho^\theta})$ hence we have

$$\begin{aligned}&1+O(\frac{1}{\rho^\theta})+\frac{1}{2}\cos\pi\mu_3\, e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt}-\frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt+\frac{i\pi}{2}} \\ &=\cos\pi\mu_3\, e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt+\frac{i\pi}{2}}+\sin^2\pi\mu_3\, e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt+\frac{i\pi}{2}}\end{aligned} \quad (39)$$

$$K+O(\frac{1}{\rho^\theta})=e^{-2i\rho\int_{x_2}^{x}|\phi(t)|dt+\frac{i\pi}{2}}(\cos\pi\,\mu_3+\sin^2\pi\,\mu_3), \quad (40)$$

Such that K is a fixed number and are given in the following form

$$\begin{aligned}&K=1+\frac{1}{2}\cos\pi\,\mu_3\, e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt}-\frac{1}{2}\cos^2\pi\mu_3 e^{-2i\rho\int_{x_2}^{x_3}|\phi(t)|dt+\frac{i\pi}{2}} \\ &\frac{K}{\cos\pi\,\mu_3+\sin^2\pi\,\mu_3}+O(\frac{1}{\rho^\theta})=e^{-2i\rho\int_{x_2}^{x}|\phi(t)|dt+\frac{i\pi}{2}}\end{aligned} \quad (41)$$

$$L+O(\frac{1}{\rho^\theta})=e^{-2i\rho\int_{x_3}^{x}|\phi(t)|dt+\frac{i\pi}{2}+2k\pi i}. \quad (42)$$

Here we called L = $\dfrac{K}{\cos\pi\,\mu_3+\sin^2\pi\,\mu_3}$

$$O(\frac{1}{\rho^\theta})=-2i\rho\int_{x_3}^{x}\left|\phi(t)\right|dt+\frac{i\pi}{2}+2k\pi i \tag{43}$$

by factoring -2i, (43) can be written as follows

$$O(\frac{1}{\rho^\theta})=\rho\int_{x_3}^{x}\left|\phi(t)\right|dt-\frac{\pi}{4}-k\pi \tag{44}$$

where

$$\rho=\frac{k\pi+\frac{\pi}{4}}{\int_{x_3}^{x}\left|\phi(t)\right|dt}+O(\frac{1}{k^\theta}), \tag{45}$$

since ρ = O(k), then the eigenvalues of equation (1) are obtained in the following form

$$\rho_k=\frac{k\pi+\frac{\pi}{4}}{\int_{x_3}^{x}\left|\phi(t)\right|dt}+O(\frac{1}{k^\theta}) \tag{46}$$

REFERENCES

Abramowitz, M., J.A. Stegun, 1964. Handbook of Mathematical functions, Appl. Math. Ser.55, U. S. Govt. Printing Office, D.C. Washington,

Eberhard, W., G. Freiling, A. Schneider, 1994. Connection formula for second order differential equations with a complex parameter and having an arbitrary number of turning points, Math. Nachr. 165: 205-229.

Dyachenko, A.X., 2000. Asymptotics of the eigenvalues of an indefinite Sturm-Liouville problem, Math. Notes 68(I): 120-124.

Jodayree Akbarfam A., 1997. Higher-order asymptotice approximatios to the eigenvalues of the Sturm -liouville problems in one turning point case, Bulletin Iranian Math. Society, 23(2): 37-53.

Leung, A., 1997. Distribution of eigenvalues in the presence of higher order turning points, Trans. Amer. Math. Soc. 229: 111-135.

Neamaty A. and E.A. Sazgar, 2009. The Neumann conditions for Sturm-Liouville problems with turning points , Int. J. Contemp. Math. Sciences. 3(12): 551-559.

Neamaty A. and E. A. Sazgar, 2009. The Negative Neumann Eigenvalues of Second Order Differential Equation with Two Turning Points, Applied Mathematical Sciences, 3(2): 61- 66.

Olver, F.W.J., 1997. Connection formula for second order differential equations having an arbitrary number of turning points of arbitrary multiplicities, SIAM J. Math, Anal. 8: 673-700.

Olver, F.W.J., 1997. Connection formula for second order differential equations hwith multiple turning points, SIAM J. Math, Anal. 8: 127-154.