Australian Journal of Basic and Applied Sciences, 5(5): 1338-1345, 2011
ISSN 1991-8178

Approximate Solution of Integro-differential Equations System

M. Zarebnia

Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran.

Abstract: In this paper, a numerical procedure for solving a class of system of Fredholm integro-differential equations, using the globally defined Sinc basis functions is considered. Properties of the Sinc procedure are utilized to reduce the computation of the integro-differential equations system to some algebraic equations. To illustrate the accuracy and the implementation of the method some numerical examples are used.

Key words: Integro-differential; Fredholm; System; Sinc.

INTRODUCTION

Consider the system of linear Fredholm integro-differential equation of the form:

$$Y'(x) = \int_a^b K(x,t)Y(t)dt + \tilde{\sigma}(x)Y(x) + G(x), \quad a \le x \le b, \qquad (1)$$
$$Y(a) = Y_a,$$

where

$$Y(x) = [y_1(x), y_2(x), \ldots, y_n(x)]^T,$$
$$\tilde{\sigma}(x) = [\sigma_{i,j}(x)],\ i, j = 1,2,\ldots,n,$$
$$G(x) = [g_1(x), g_2(x), \ldots, g_n(x)]^T,$$
$$K(x,t) = [k_{i,j}(x,t)],\ i, j = 1,2,\ldots,n,$$
$$Y(a) = [y_1(a), y_2(a), \ldots, y_n(a)]^T.$$

In system (1) Y (x) is the solution to be determined and the kernel K (x,t), functions G (x) and σ (x) are given. In recent years, many different methods have been used to estimate the solution of integro-differential equations and systems (Arikoglu *et al.*, 2008, Avudainayagam *et al.*, 2000, Jackiewicz *et al.*, 2008, Yusufoglu, 2007, 2009). In this paper a global approximation (Stenger, 1993, Lund *et al.*, 1992) for the solution of the Eq. (1) using the Sinc functions is developed. Our method consists of reducing the solution of (1) to a set of algebraic equations. The properties of Sinc function are then utilized to evaluate the unknown coefficients.

The outline of the paper is as follows. First, in section 2 we review some of the main properties of Sinc function and Sinc method that are necessary for the formulation of the discrete system. In section 3, we illustrate how the Sinc method may be used to replace Eq. (1) by an explicit system of linear algebraic equations. In section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

Corresponding Author: M. Zarebnia, Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran.
Tel: +98 (451) 5514702
E-mail: zarebnia@uma.ac.ir

2. *Properties of Sinc Functions:*

The Sinc function properties and the Sinc method are discussed thoroughly by Stenger, 1993 and Lund *et al.*, 1992. For any h > 0, the Sinc basis functions are given by

$$S(j,h) = \sin c\left(\frac{z - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where

$$\sin c(z) = \begin{cases} \dfrac{\sin(\pi z)}{\pi z}, z \neq 0; \\ 1, \qquad z = 0. \end{cases} \tag{3}$$

The Sinc function for the interpolating points $Z_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j; \\ 0, & k \neq j. \end{cases} \tag{4}$$

Let

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} dt, \tag{5}$$

then define a matrix $I^{(-1)} = \left[\delta_{kj}^{(-1)}\right]$ whose (k,j) th entry is given by $\delta_{kj}^{(-1)}$. They are based in the infinite strip D_d in the complex plane

$$D_d = \{w = u + iv : |v| < d \leq \frac{\pi}{2}\}. \tag{6}$$

To construct approximation on the interval (a ,b) we consider the conformal map

$$\phi(z) = \ln\left(\frac{z - a}{b - z}\right). \tag{7}$$

The map ϕ carries the eye-shaped region

$$D = \left\{ z = x + iy : |\arg(\frac{z - a}{b - z}) < d \leq \frac{\pi}{2} \right\} \tag{8}$$

The function

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w} \tag{9}$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D : -\infty < u < \infty\}. \quad (10)$$

The Sinc grid points $z_k \in (a,b)$ in *D* will be denoted by x_k because they are real. For the evenly spaced $\{kh\}_{k=-\infty}^{\infty}$ nodes on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}},\ k = \pm 1, \pm 2, \ldots. \quad (11)$$

Definition 1:
A function *y (z)* is in the space $L_a(D)$ if and only if *y (z)* is analytic in *D* and there exists a constant, C > 0, such that

$$| y(z) | \le C \frac{| \rho(z) |^{\alpha}}{[1 + | \rho(z) |]^{2\alpha}},\ z \in D,\ 0 < \alpha \le 1. \quad (12)$$

Theorem 1:
Let $\frac{y}{\phi'} \in L_{\alpha}(D)$, let $\delta_{kj}^{(-1)}$ be defined as in (5), and let $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$. Then there exists a constant, C_1, which is independent of *N*, such that

$$\left| \int_a^{z_k} y(t)dt - h \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{y(z_j)}{\phi'(z_j)} \right| \le c_1 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \quad (13)$$

Theorem 2:
Let $\frac{y}{\phi'} \in L_{\alpha}(D)$, let *N* be a positive integer and let $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$, then there exist positive constant, c_2, independent of *N*, such that

$$\left| \int_{\Gamma} y(z)dz - h \sum_{j=-N}^{N} \frac{y(z_j)}{\phi'(z_j)} \right| \le c_2 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \quad (14)$$

3. *The Approximate Solution of System of Fredholm Integro-differential Equations:*

Let us consider the system of linear integro-differential equations (1). Consider the *i* th equation of (1):

$$y_i'(x) = \sum_{j=1}^{n} \int_a^b K_{i,j}(x,t) y_j(t)dt + \sum_{j=1}^{n} \sigma_{i,j}(x) y_j(x) + g_i(x),\ i = 1, 2, \ldots, n. \quad (15)$$

By integrating from *a* to x, we obtain:

$$y_i(x)=\int_a^x\left\{\sum_{j=1}^{n}\int_a^b K_{i,j}(\xi,t)y_j(t)dt+\sum_{j=1}^{n}\sigma_{i,j}(\xi)y_j(\xi)+g_i(\xi)\right\}d\xi+y_i(a),\quad i=1,2,\dots,n. \tag{16}$$

For simplicity, we write

$$F_{i,j}(\xi)=\int_a^b K_{i,j}(\xi,t)y_j(t)dt. \tag{17}$$

Assume that $\frac{F_{i,j}}{\phi'}\in L_\alpha(D)$, $\frac{\sigma_{i,j}}{\phi'}y_j\in L_\alpha(D)$ and $\frac{g_i}{\phi'}\in L_\alpha(D)$: By setting $.x=x_k,\ \ k=-N,\dots,N$ considering Theorem 1 for the right-hand side of (16), and applying the collocation to it, we obtain the following equations;

$$\begin{aligned}y_i(x_k)&=h\sum_{j=1}^{n}\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{F_{i,j}(x_l)}{\phi'(x_l)}+h\sum_{j=1}^{n}\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{\sigma_{i,j}(x_l)}{\phi'(x_l)}y_j(x_l)\\&+h\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{g_i(x_l)}{\phi'(x_l)}+y_i(a),\ i=1,2,\dots,n,\ k=-N,\dots N.\end{aligned} \tag{18}$$

Using Theorem 2 for the first term on the right-hand side of the above relation, we get;

$$h\sum_{j=1}^{n}\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{F_{i,j}(x_l)}{\phi'(x_l)}\approx h^2\sum_{j=1}^{n}\sum_{l=-N}^{N}\sum_{l'=-N}^{N}\delta_{k,l}^{(-1)}\frac{K_{i,j}(x_l,t_{l'})}{\phi'(x_l)\phi'(t_{l'})}y_j(t_{l'}). \tag{19}$$

Having replaced the first term on the right-hand side of (18) by equation (19), we have;

$$\begin{aligned}y_i(x_k)&=h^2\sum_{j=1}^{n}\sum_{l=-N}^{N}\sum_{l'=-N}^{N}\delta_{k,l}^{(-1)}\frac{K_{i,j}(x_l,t_{l'})}{\phi'(x_l)\phi'(t_{l'})}y_j(t_{l'})+h\sum_{j=1}^{n}\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{\sigma_{i,j}(x_l)}{\phi'(x_l)}y_j(x_l)\\&+h\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{g_i(x_l)}{\phi'(x_l)}+y_i(a),\ i=1,2,\dots,n,\ k=-N,\dots,N,\end{aligned} \tag{20}$$

where

$$\phi(x)=\ln\left(\frac{x-a}{b-x}\right),\ \phi(a)=-\infty,\ \phi(b)=+\infty,\ \phi'(x)=\frac{a-b}{(a-x)(b-x)}.$$

The equation (20) is rewritten as follows:

$$\begin{aligned}&y_{i,k}-h^2\sum_{j=1}^{n}\left[\sum_{l=-N}^{N}\sum_{l'=-N}^{N}\delta_{k,l}^{(-1)}\frac{K_{i,j}(x_l,t_{l'})}{\phi'(x_l)\phi'(t_{l'})}y_{j,l'}\right]-h\sum_{j=1}^{n}\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{\sigma_{i,j}(x_l)}{\phi'(x_l)}y_{j,l}\\&\approx h\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{g_i(x_l)}{\phi'(x_l)}+y_i(a),\ i=1,2,\dots,n,\ k=-N,\dots,N.\end{aligned} \tag{21}$$

The above equation is a system of $n \times (2N+1)$ linear equations with $n \times (2N+1)$ unknowns $y_{i,k},\ k=-N,\dots,N,\ i=1,2,\dots,n$ where $y_{i,k}$ denotes an approximate value of $y_i(x_k)$ Denote

$$I^{(m)}=[\delta_{k,j}^{(m)}],\quad m=-1,0,1,\quad D(\sigma_{i,j}/\phi')=diag(\sigma_{i,j}(x_{-N})/\phi'(x_{-N}),\dots,\sigma_{i,j}(x_N)/\phi'(x_N))$$

$$\tilde{K}_{i,j}=\left[\frac{K_{i,j}(x_l,t_{l'})}{\phi'(x_l)\phi'(x_{l'})}\right]\text{and}$$

$$B_{i,j}=\begin{cases}I^{(0)}-h^2I^{(-1)}\tilde{K}_{i,j}-hI^{(-1)}D(\sigma_{i,j}/\phi'), & i=j;\\ -h^2I^{(-1)}\tilde{K}_{i,j}-hI^{(-1)}D(\sigma_{i,j}/\phi'), & i\neq j.\end{cases}$$

The system of linear equations (21) for $n \times (2N+1)$ unknown coefficients $y_{j,l},\ j=1,2,\dots,n,\ l=-N,\dots,N,$ can be expressed in a matrix form

$$B\tilde{Y}=\tilde{G}, \tag{22}$$

where

$$B=\begin{pmatrix}B_{1,1} & B_{1,2} & \dots & B_{1,n}\\ B_{2,1} & B_{2,2} & \dots & B_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ B_{n,1} & B_{n,2} & \dots & B_{n,n}\end{pmatrix},$$

$$\tilde{G}=[R_{1,-N},\dots,R_{1,N},\dots,R_{n,-N},\dots,R_{n,N}]^T,$$

$$R_{i,k}=h\sum_{l=-N}^{N}\delta_{k,l}^{(-1)}\frac{g_i(x_l)}{\phi'(x_l)}+y_i(a),$$

$$\tilde{Y}=[y_{1,l},\dots,y_{n,l}]^T,\ l=-N,\dots,N.$$

The approximate solution $y_{j,l},\ l=-N,\dots,N,\ j=1,2,\dots,n$ is obtained by solving the system (21). Then by using the $y_{j,l},\ l=-N,\dots,N,\ j=1,2,\dots,n$ and employing a method similar to the Nystrom's idea we can obtain an approximation to the solution of system (1):

$$Y_N(x)=h^2\sum_{l=-N}^{N}\sum_{j=-N}^{N}\frac{K(\xi_l,t_j)}{\phi'(x_l)\phi'(t_j)}\Omega_{h,l}(x)\tilde{Y}_j+h\sum_{l=-N}^{N}\frac{\tilde{\sigma}(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)\tilde{Y}_l$$
$$+h\sum_{l=-N}^{N}\frac{G(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)+Y(a), \tag{23}$$

where

$$\tilde{Y}_l = [y_{1,l}, y_{2,l}, \ldots, y_{n,l}]^T,$$
$$K(\xi_l, t_j) = [k_{i,j'}(x,t)],\ i, j' = 1,2,\ldots,n,$$
$$\tilde{\sigma}(\xi_l) = [\sigma_{i,j}(\xi_l)],\ i, j = 1,2,\ldots,n,$$
$$G(\xi_l) = [g_1(\xi_l), g_2(\xi_l), \ldots, g_n(\xi_l)]^T,$$
$$Y(a) = [y_1(a), y_2(a), \ldots, y_n(a)]^T,$$
$$\Omega_{h,l}(x) = \frac{1}{2} + \int_a^x S(l,h) \circ \phi(t)dt.$$

4. *Numerical Examples:*

In order to illustrate the performance of the Sinc method in solving integro-differential equations system and justify the accuracy and efficiency of the presented method, we consider the following examples. The examples have been solved by presented method with different values of *N*. In all examples we take $\alpha = \frac{1}{2}$ and $d = \frac{\pi}{2}$ which yield $h = \pi\left(\frac{1}{N}\right)^{\frac{1}{2}}$. The errors are reported on the set of Sinc grid points

$$S = \{x_{-N}, \ldots, \chi_o, \ldots, x_N\},$$
$$x_k = \frac{e^{kh}}{1+e^{kh}},\ k = -N, \ldots, N_o, \tag{24}$$

The maximum absolute error on the Sinc grid points is

$$\left\| E_Y^S(h) \right\|_\infty = \max_{-N \le j \le N} \left| Y(x_j) - Y_N(x_j) \right|. \tag{25}$$

Example 1:

Consider the following system of Fredholm integro-differential equations with exact solution

$(y_1(x), y_2(x)) = (3x^2+1, x^3+2x-1)$:

$$\begin{cases} y_1'(x) = -\int_0^1 3(2x+t^2)y_1(t)dt + \int_0^1 6(2x+t^2)y_2(t)dt + 15x + \frac{4}{5}, \\ y_2'(x) = -\int_0^1 2xty_1(t)dt + \int_0^1 6xty_2(t)dt + 3x^2 + \frac{3}{10}x + 2, \\ y_1(0) = 1,\ y_2(0) = -1. \end{cases} \tag{26}$$

Example 1 is solved for different values of *N*. The maximum of absolute errors on the Sinc grid *S* are tabulated in Table 1.

Example 2:
We consider the system of integro-differential equations

$$\begin{cases} y_1'(x) = -\int_0^1 e^{(x-t)} y_1(t)dt + \int_0^1 e^{(x+2)t} y_2(t)dt + y_1(x) + g_1(x), \\ y_2'(x) = -\int_0^1 \cos(4\pi x)\sin(2\pi t) y_1(t)dt - \int_0^1 \cos(4\pi x + 2\pi t) y_2(t)dt + y_1(x) + y_2(x) + g_2(x), \\ y_1(0) = 1,\ y_2(0) = 1, \end{cases} \quad (27)$$

where g_1 (x) and g_2 (x) are chosen such that the exact solution is $(y_1(x), y_2(x)) = (e^x, e^{-x})$.

The approximate solutions are calculated for different values of *N* and the optimal Sinc mesh size

$h = \pi \left(\frac{1}{N}\right)^{\frac{1}{2}}$. Table 2 exhibits the absolute errors.

Table 1: Results for Example 1.

N	*h*	$\left\|E_{y_1}^S(h)\right\|_\infty$	$\left\|E_{y_2}^S(h)\right\|_\infty$
5	1.404963	3.58412×10^{-2}	1.81362×10^{-2}
10	0.9963459	4.61962×10^{-4}	2.47935×10^{-4}
20	0.702481	8.78212×10^{-6}	3.13909×10^{-6}
30	0.573574	5.48741×10^{-7}	2.13124×10^{-7}
40	0.496729	4.31708×10^{-8}	1.69706×10^{-8}
50	0.444288	4.37749×10^{-9}	1.73758×10^{-9}
60	0.405578	5.42173×10^{-10}	2.18473×10^{-10}
70	0.375492	7.89298×10^{-11}	3.22988×10^{-11}
80	0.351241	1.30291×10^{-11}	5.44509×10^{-11}
90	0.331153	2.73899×10^{-12}	4.71345×10^{-12}
100	0.314159	4.87610×10^{-13}	2.15161×10^{-13}

Table 1: Results for Example 2.

N	*h*	$\left\|E_{y_1}^S(h)\right\|_\infty$	$\left\|E_{y_2}^S(h)\right\|_\infty$
5	1.404963	2.78724×10^{-3}	2.73763×10^{-3}
10	0.993459	2.71654×10^{-4}	2.03569×10^{-4}
20	0.702481	5.70513×10^{-6}	4.71787×10^{-6}
30	0.573574	2.85487×10^{-7}	2.44982×10^{-7}
40	0.496729	2.24748×10^{-8}	1.96874×10^{-8}
50	0.444288	2.36912×10^{-9}	2.10371×10^{-9}
60	0.405578	3.07697×10^{-10}	2.75940×10^{-10}
70	0.375492	4.68438×10^{-11}	4.23366×10^{-11}
80	0.351241	8.09139×10^{-12}	7.36045×10^{-12}
90	0.331153	1.55559×10^{-12}	1.41712×10^{-12}
100	0.314159	3.46889×10^{-13}	2.97484×10^{-13}

REFERENCES

Arikoglu, A., I. Ozkol, 2008. Solutions of integral and integro-differential equation systems by using differential transform method. Computers and Mathematics with Applications, 56: 2411-2417.

Avudainayagam, A., C. Vani, 2000. Wavelet-Galerkin method for integrodifferential equations. Appl. Numer. Math., 32: 247-254.

Jackiewicz, Z., M. Rahman, B. D.Welfert, 2008. Numerical solution of a Fredholm integro-differential equation modelling neural networks. Appl. Numer. Math., 195: 523-536.

Lund, J., K. Bowers, 1992. Sinc Methods For Quadrature and Differential Equations. SIAM, Philadelphia, PA.

Stenger, F., 1993. Numerical methods based on Sinc and analytic functions. Springer-Verlag, New York.

Yusufoglu, E., 2007. An efficient algorithm for solving integro-differential equations System. Applied Mathematics and Computation, 192: 51-55.

Yusufoglu, E., 2009. Numerical solving initial value problem for Fredholm type linear integro-differential equation system. Journal of the Franklin Institute, 346: 636-649.