The Solution Some Equations by Perturbation Method

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Abstract: Perturbation method is an efficient method for obtaining solutions of Nonlinear partial differential equations. In this paper, we look for the exact solutions of some equations via perturbation method. 2000 Mathematics Subject Classification: 35J60, 35A05.

Key words: Perturbation method; Klein-Gordon equation; Fisher equation; Huxley equation; Nonlinear partial differential equation.

INTRODUCTION

The study of numerical methods for the solution of nonlinear partial differential equations (NLPDEs) has enjoyed an intense period of activity over the last 40 years from both theoretical points of view. Improvements in numerical techniques, together with the rapid advances in computer technology, have meant that many of the partial differential equations arising from engineering and scientific applications, which were previously intractable, can now, be routinely solved (Mitchell and Griffiths, 1980). A variety of powerful methods, such as the homogeneous balance method (Wang, 1995; Fan and Zhang, 1998), the hyperbolic tangent function expansion method (Parkes and Duffy, 1997; Zhang, 1999), the trial function method (Kudryshow, 1990; Xie and Tang, 2006), the sine-cosine method (C.T.Yan, 1996), the Jacobi elliptic function expansion method (Liu, et al 2001; Z.T fu, et al 2003), the superposition method (Xie, and J.S Tang, 2005), and so on, were used to investigate nonlinear dispersive and dissipative problems.

Our first interest in present work begin in implementing the perturbation method do stress its power in handling nonlinear equations so that one can apply it to models of various types of nonlinearity. The aim of this paper is to find solution of the Klein-Gordon equation, Fisher equation and Huxley equation.

1. The Perturbation Method:

Let us consider given equation in the following form

\[ F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \]

(1)

If we can be written Eq. (1) as follows:

\[ L^\wedge (D_t, D_x)u(t, x) = N[u], \]

(2)

where

\[ L^\wedge (D_t, D_x)u(t, x) = \sum_{k=0}^{K} \sum_{m=0}^{M} l_{km} D_t^k D_x^m, \]

(3)

is linear operator of Eq. (1) and N[u] is nonlinear term of Eq. (1), then, we look for an exact solution of Eq. (2), in the form

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Substituting (4) into Eq. (2), and collecting like powers of \( \varepsilon \) we get series of equations. Thus, we can obtain all the \( \{u_i\}_{i=1}^{n} \). Setting \( u_i' \) in (4), we get the exact solution of Eq. (1).

Remark. Here \( \varepsilon \) is the grading parameter, finally we can put \( \varepsilon = 1 \).

2. The Klein-gordon Equation in Special Case:

Let us consider the Klein-Gordon equation in special case \( (a = b = c = 1) \)

\[
u_{tt} - u_{xx} + u = u^3
\]  

Thus, we can be written

\[
L(D_t, D_x)u(t, x) = u^3
\]  

\[
\hat{L}(D_1, D_X) = D_1^2 - D_X^2 + 1.
\]

Now, we look for a solution of (6) in the form (4). Substituting (4) into (6) and collecting equal powers of \( \varepsilon \) we obtain the following series equations:

\[
\hat{L}u_1 = 0,
\]

\[
\hat{L}u_2 = 0,
\]

\[
\hat{L}u_3 = u_1^3,
\]

\[
\hat{L}u_4 = 3u_1^2u_2,
\]

\[
\hat{L}u_5 = 3u_1^2u_3 + 3u_1u_2^2,
\]

\[
\hat{L}u_6 = 3u_1^2u_4 + 6u_1u_2u_3 + u_3^3,
\]

\[
\hat{L}u_7 = 6u_1u_2u_4 + 3u_1u_3^2 + 3u_2^2u_3 + 3u_1^2u_5,
\]

\[
\vdots
\]

\[
\hat{L}u_n = \sum_{k=2}^{n-1} u_{n-k} \sum_{l=1}^{k-1} u_l u_{k-l}, n \geq 3
\]

\[
\vdots
\]

By using \( Lu_1 = 0 \), gives

\[
u_1 = a_1 e^{\sqrt{3}x-t}
\]

where \( a_1 \) is real constant.

From \( \hat{L}u_2 = 0 \), we conclude \( u_2 = 0 \), thus all coefficients of \( u_2 \) will be zero. means:

\[
u_{2p+2} = 0, p = 0, 1, 2, \ldots
\]
We have

\[ L u_3 = u_1^3. \]  

(9)

Setting (7) into (9), we obtain

\[ u_3 = \frac{a_1^3}{8} e^{\sqrt{2}\sqrt{x-3}t} \]  

(10)

\[ L u_5 = 3u_1^2u_3 + 3u_1u_2^2. \]  

(11)

Applying (8) and substituting (7) and (10) into (11), gives

\[ u_5 = \frac{a_1^5}{8^2} e^{\sqrt{2}\sqrt{x-5}t} \]  

(12)

\[ L u_7 = 6u_1u_2u_4 + 3u_1u_3^2 + 3u_2u_3^2 + 3u_1u_5. \]  

(13)

By using (8) and substituting (7) and (10) and (12) into (13), we have solution take the form

\[ u_7 = -\frac{a_1^7}{8^3} e^{\sqrt{2}\sqrt{x-7}t} \]  

(14)

\[ \vdots \]

Same above, we can calculate all \( u_{2n+1} \) for \( n \geq 4 \). Applying (4), we obtain

\[ u(x,t) = \sum_{n=1}^{\infty} \left( -\frac{1}{8} \right)^n \left( e a_1 e^{\sqrt{2}\sqrt{x-t}} \right)^{2n-1} = \frac{e a_1 e^{\sqrt{2}\sqrt{x-t}}}{1 + \frac{e^2 a_1^2 e^{2\sqrt{2}\sqrt{x-t}}}{8}} \]  

(15)

Let

\[ \frac{a_1^2}{8} = 1. \]  

(16)

Substituting (16) into (15), concludes

\[ u(x,t) = \pm \sqrt{8} \sec h(\sqrt{2}x-t) \]  

(17)

Therefore, the (17) is a exact soliton solution of the Klein-Gordon equation in special case (\( a = b = c = 1 \)).

3. The Klein-gordon Equation in the General Case:

Let us consider the Klein-Gordon equation in the general case

\[ u_{tt} - au_{xx} + bu = cu^3 \]  

(18)

Therefore, we can be written
\( L(\partial_t, \partial_x)u(t, x) = cu^3 \) \hfill (19)

\( L(\partial_t, \partial_x) = D_t^2 - aD_x^2 + b. \)

Now we seek a solution of (19) in the form (4). Substituting (4) into (19) and collecting equal powers of \( \varepsilon \), we obtain the following series equations:

\[
\begin{align*}
\hat{L}u_1 &= 0, \\
\hat{L}u_2 &= 0, \\
\hat{L}u_3 &= cu_1^3, \\
\hat{L}u_4 &= 3cu_1^2u_2, \\
\hat{L}u_5 &= 3cu_1^2u_3 + 3cu_2u_2^2, \\
\hat{L}u_6 &= 3cu_1^2u_4 + 6cu_1u_2u_3 + cu_2^3, \\
\hat{L}u_7 &= 6cu_1u_2u_4 + 3cu_2u_3 + 3cu_2u_3^2 + 3cu_3u_5, \\
\vdots \\
\hat{L}u_n &= c \sum_{k=2}^{n-1} u_{n-k} \sum_{l=1}^{k-1} u_l u_{k-l}, n \geq 3
\end{align*}
\]

By using \( \hat{L}u_1 = 0 \), gives

\[
u_1 = a_1 e^{x - \sqrt{u-b}t} \tag{20}\]

where \( a_1 \) is real constant.

From \( \hat{L}u_2 = 0 \), we conclude \( u_2 = 0 \), thus all coefficients of \( u_2 \) will be zero. means:

\[
u_{2p+2} = 0, p = 0, 1, 2, \ldots \tag{21}\]

We have

\[
\hat{L}u_3 = cu_1^3. \tag{22}
\]

Setting (7) into (9), we obtain

\[
u_3 = \frac{a_1^3}{-8b} e^{3x - 3\sqrt{u-b}t}. \tag{23}\]

\[
\hat{L}u_5 = 3cu_1^2u_3 + 3cu_2u_2^2. \tag{24}\]

Applying (21) and setting (20) and (23) into (24), gives
\[ u_5 = \frac{c^2}{8b^2} t^5 e^{5x-5\sqrt{a-bt}}. \]  
\[ (25) \]

\[ L^t u_7 = 6cu_t u_4 + 3cu_1 u_4^2 + 3cu_2 u_3 + 3cu_1^2 u_5. \]  
\[ (26) \]

By using (21) and substituting (20) and (23) and (25) into (26), we have

\[ u_7 = \frac{c^3}{-8^2 b^3} t^7 e^{7x-7\sqrt{a-bt}}. \]  
\[ (27) \]

Same above, we can calculate all \( u_{2n+1} \) for \( n \geq 4 \). Applying (4), we obtain

\[ u(x,t) = \sum_{n=1}^{\infty} \left( -\frac{c}{8b} \right)^{n-1} \left( \varepsilon a_1 e^{x-\sqrt{a-bt}} \right)^{2n-1} = \frac{\varepsilon a_1 e^{x-\sqrt{a-bt}}}{1 + \varepsilon^2 a_1^2 c e^{2x-2\sqrt{a-bt}}} \]  
\[ (28) \]

Let

\[ \frac{ca_1^2}{8b} = 1. \]  
\[ (29) \]

Substituting (29) into (28), concludes

\[ u(x,t) = \pm \sqrt{\frac{2b}{c}} \sec h(x - \sqrt{a - bt}) \]  
\[ (30) \]

Therefore, the (30) is a exact soliton solution of the Klein-Gordon equation.

4. The Fisher Equation:

Let us consider the Fisher equation

\[ u_t - u_{xx} - u - u^2 = 0 \]  
\[ (31) \]

Thus, we can be written

\[ L^D (D_t, D_x) u(t,x) = -u^2 \]  
\[ (32) \]

\[ L^\varepsilon (D_t, D_x) = D_t - D_x^2 - 1. \]

Now, we look for a solution of (32) in the form (4). Substituting (4) into (32) and collecting equal powers of \( \varepsilon \) we obtain the following series equations:

\[ L^\varepsilon u_1 = 0, \]

\[ L^\varepsilon u_2 = -u_1^2, \]
\[ L^\ast u_3 = -2u_1u_2, \]
\[ L^\ast u_4 = -2u_1u_3 - u_2^2, \]
\[ \vdots \]
\[ L^\ast u_n = -\sum_{k=1}^{n-1} u_k u_{n-k}, \]

By using \( L^\ast u_1 = 0\), gives

\[ u_1 = a_1 e^{\sqrt{x+2(l+1)t}} \quad (33) \]

where \( a_1 \) is real constant.

We have

\[ L^\ast u_2 = -u_1^2. \quad (34) \]

Setting (33) into (34), we obtain

\[ u_2 = \frac{-a_1^2}{-2l+1} e^{2\sqrt{x+2(l+1)t}} \quad (35) \]

\[ L^\ast u_3 = -2u_1u_2. \quad (36) \]

Setting (33) and (35) into (36), we have

\[ u_3 = \frac{a_1^3}{(-3l+1)(-2l+1)} e^{3\sqrt{x+3(l+1)t}} \quad (37) \]

\[ L^\ast u_4 = -2u_1u_3 - u_2^2. \quad (38) \]

By substituting (33) and (35) and (37) into (38), we have solution take the form

\[ u_4 = \frac{(4l-3)a_1^4}{3(-4l+1)(-3l+1)(-2l+1)^2} e^{4\sqrt{x+4(l+1)t}} \quad (39) \]

\[ \vdots \]

Applying (4), we obtain

\[ u(x,t) = e a_1 e^{\sqrt{x+2(l+1)t}} + e^2 \frac{-a_1^2}{-2l+1} e^{2\sqrt{x+2(l+1)t}} + e^3 \frac{a_1^3}{(-3l+1)(-2l+1)} e^{3\sqrt{x+3(l+1)t}} \]

\[ + e^4 \frac{(4l-3)a_1^4}{3(-4l+1)(-3l+1)(-2l+1)^2} e^{4\sqrt{x+4(l+1)t}} + \ldots \quad (40) \]
5. The Huxley Equation:

Let us consider the Huxley equation

$$u_t - u_{xx} - (a + 1)u^2 + u^3 + au = 0, \quad (41)$$

Therefore, we can be written

$$L^*(D_t, D_x)u(t, x) = -u^3 + (a + 1)u^2, \quad (42)$$

$$L^*(D_t, D_x) = D_t - D_x^2 + a.$$ 

Now we seek a solution of (42) in the form (4). Substituting (4) into (42) and collecting equal powers of $\varepsilon$, we obtain the following series equations:

$$L^* u_1 = 0,$$

$$L^* u_2 = (a + 1)u_1^2,$$

$$L^* u_3 = -u_1^3 + 2(a + 1)u_1u_2,$$

$$u_3 = \frac{2a^2 + 5a + 4}{(-4a - 6)(-2 - a)} a_1^3 e^{3x + 3(1-a)t}.$$ 

$$L^* u_4 = -3u_1^2u_2 + (a + 1)(u_2^2 + 2u_3),$$

$$u_4 = \frac{-4a^4 - 52a^3 - 134a^2 - 144a - 58}{(-5a - 12)(-2 - a)^2(-4a - 6)} a_1^4 e^{4x + 4(1-a)t}.$$ 

$$u(x, t) = \varepsilon a_1 e^{x(1-a)t} + \varepsilon^2 \frac{a + 1}{-2 - a} a_1^2 e^{2x + 2(1-a)t} + \varepsilon^3 \frac{2a^2 + 5a + 4}{(-4a - 6)(-2 - a)} a_1^3 e^{3x + 3(1-a)t} + \varepsilon^4 \frac{-4a^4 - 52a^3 - 134a^2 - 144a - 58}{(-5a - 12)(-2 - a)^2(-4a - 6)} a_1^4 e^{4x + 4(1-a)t} + \ldots$$ 

$$L^* u_5 = -u_1^4 + 2(a + 1)u_1u_2,$$

$$L^* u_4 = -3u_1^2u_2 + (a + 1)(u_2^2 + 2u_3),$$

$$\vdots$$

$$L^* u_n = -\sum_{k=2}^{n-1} \sum_{l=1}^{n-1} u_{n-k}u_{l} + (a + 1)\sum_{k=1}^{n-1} u_ku_{n-k}n \geq 2$$

By using $L^* u_1 = 0$, gives

$$u_1 = a_1 e^{x(1-a)t} \quad (43)$$

Where $a_1$ and $a$ are real constants

We have
\[ \dot{u}_2 = (a + 1)u_1^2. \]  
(44)

Setting (43) into (44), we obtain
\[ u_2 = \frac{a + 1}{-2 - a} a_1^2 e^{2x + 2(1-a)t}. \]  
(45)

\[ \dot{u}_3 = -u_1^2 + 2(a + 1)u_1u_2. \]  
(46)

Setting (43) and (45) into (46), gives
\[ u_3 = \frac{2a^2 + 5a + 4}{(-4a - 6)(-2 - a)} a_1^3 e^{3x + 3(1-a)t}. \]  
(47)

\[ \dot{u}_4 = -3u_1^2u_2 + (a + 1)(u_2^2 + 2u_1u_3). \]  
(48)

By substituting (43) and (45) and (47) into (48) we have solution take the form
\[ u_4 = \frac{-4a^4 - 52a^3 - 134a^2 - 144a - 58}{(-5a - 12)(-2 - a)^2(-4a - 6)} a_1^4 e^{4x + 4(1-a)t}. \]  
(49)

Applying (4), we obtain
\[ u(x, t) = e_a e^{x + (1-a)t} + e^{2} \frac{a + 1}{-2 - a} a_1^2 e^{2x + 2(1-a)t} + e^{3} \frac{2a^2 + 5a + 4}{(-4a - 6)(-2 - a)} a_1^3 e^{3x + 3(1-a)t} + \ldots \]  
(50)

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