Kalman Filter for Estimate the State of Shallow Water Model

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Abstract: In this paper, the linearized one-dimensional shallow water model is used as the mathematical model for a fluid wave with no mean velocity. The linear state space model of shallow water equation is constructed by linearizing and discretizing with finite difference method. The Kalman filtering technique is used to estimate the state of linear state space model by incorporating the measured data. Then, the filtering process is implemented in order to remove the noise from both measured data and finite difference approximation. Finally, the performance of Kalman filter for estimating the state of model is presented.

Key words: Kalman filter; Shallow water model; Finite difference method

INTRODUCTION

Shallow water model is widely used to model waves in the atmosphere, rivers, lakes and oceans. For shallow water equations to be valid, the wave length of the phenomenon they are supposed to model has to be much larger than the depth where the phenomenon takes place. Shallow water equations are especially suitable to model waves which have very large length scale. However, since waves in fluid are fluctuated in time and space; caused by fluid velocity and fluid depth, etc. Thus, a shallow water model which takes into account these effects would be needed for estimating the state of model. The problem of state estimation concerns the task of estimating the state of a system while having access only to noisy and/or inaccurate measurements from that system. This problem has been encountered in many disciplines in the field of science and engineering. The Kalman filtering technique (Kalman, R.E., 1960) is a method that can estimate the present state and predict the future state of a linear system. The Kalman filtering not only works well in practice, but it is theoretically attractive because it can remove the noise from both measured data and the model. This technique has been applied to many fields, for example estimating future demand for a product (Munroe, M., 2009) and radio communication signals corrupted with noise (Sinopoli, K.B., 2004). The state estimation using Kalman filter is used for a failure prediction method for preventive maintenance on a DC motor (Yang, S.K., T.S. Liu, 1999). The study of the steady-state performance of Kalman filters for a second-order variable gain digital phase-locked loop (DPLL) (Qian, Y., 2009). Conjunctive use of an autoregressive moving average (ARMA) model and a Kalman filter for hydrologic river routing is suggested and applied to route the 1974 flood of the Changjiang (Yangtze) River in China (Guang-Te, W., 1987). The Kalman filter is used for estimate the state of the input stimulus function (Ward, B.D. and Y. Mazaheri, 2003). In this paper the Kalman filtering technique is applied to estimate a state of shallow water model.

In this paper, the basic concept and formulation of the Kalman filtering technique is explained. This technique is used for estimate the state of linear state space model in one-dimensional shallow water equations. Then, the measurement data are incorporated into the linear state space model. After that, the filtering process is implemented in order to remove the noise. The linear state space model of shallow water equations is constructed by linearizing and discretizing with finite difference method. In Section II, the mathematical model of fluid wave is described and the linear state space model is constructed after linearizing and discretizing. In Section III, the formulas of the Kalman filtering technique are explained, which consists of two main steps. In Section IV, an application of the Kalman filtering technique to the linear state space model of shallow water equation is shown. Finally, in Section V, the conclusion of the performance of Kalman filtering technique for estimating the state of shallow water equation is presented.

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Mathematical Model:
The Shallow Water Model:

The problem of interest involves several unknown functions of space $x$ and time $t$, the governing equations for the system are the linearized one-dimensional shallow water equations (Durran, D.R., 1999).

\[
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0,
\]

where $U$ is the mean fluid velocity.

\[
\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + H \frac{\partial u}{\partial x} = 0.
\]

$u(x, t)$ is the perturbation fluid velocity.

$H$ is the mean fluid depth.

$h(x, t)$ is the perturbation fluid depth.

$g$ is the gravitational acceleration.

Consider now the simplified case in which $U=0$ and $H=1$ In this case the shallow water equations are reduced to

\[
\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} = 0,
\]

(3)

\[
\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} = 0.
\]

(4)

Discretization:
The forward in time and central in space scheme (Morton, K.W. and D.F. Mayers, 2005) is the finite difference method used for discretize (3) and (4). Let $f$ represents the dependent variables, $u$ and $h$ the derivatives become

\[
\frac{\partial f}{\partial t} = \frac{f_{i}^{k+1} - f_{i}^{k}}{\Delta t},
\]

(5)

\[
\frac{\partial f}{\partial x} = \frac{f_{i}^{k+1} - f_{i-1}^{k+1}}{2\Delta x}.
\]

(6)

Substituting these equations into (3) and (4)

\[
\frac{u_{i}^{k+1} - u_{i}^{k}}{\Delta t} + g \frac{h_{i+1}^{k} - h_{i-1}^{k}}{2\Delta x} = 0,
\]

or

\[
u_{i}^{k+1} = u_{i}^{k} - \frac{g\Delta t}{2\Delta x}(h_{i+1}^{k} - h_{i-1}^{k}),
\]

(7)

\[
\frac{h_{i}^{k+1} - h_{i}^{k}}{\Delta t} + \frac{u_{i+1}^{k} - u_{i-1}^{k}}{2\Delta x} = 0.
\]
These equations can be written in a matrix form as follows

\[
\begin{bmatrix}
    u^{k+1}_i \\
    h^{k+1}_i
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    u^k_i \\
    h^k_i
\end{bmatrix} + \begin{bmatrix}
    g\alpha & 0 \\
    0 & \alpha
\end{bmatrix} \begin{bmatrix}
    -h^k_{i+1} & h^k_i \\
    -u^k_{i+1} & u^k_i
\end{bmatrix},
\]

where \( \alpha = \frac{\Delta t}{2\Delta x} \).

In the finite difference approximations (7) and (8), it is assumed that the perturbation fluid velocity \((u)\) and the perturbation fluid depth \((h)\) are defined at the same grid points, as shown schematically in Fig. 1.

**Fig. 1:** Distribution of \(u\) and \(h\) on an unstaggered mesh.

**Linear State Space Model:**

A linear state space model is a mathematical model of a system. A state space model describes how the state propagates in time based on external influences, such as input and noise.

Using the discretization of the constitutive equations, a linear state space model is formed as follows

\[
x_{k+1} = Ax_k + Bp_k,
\]

where \( x_k = [u^k_1, u^k_2, \ldots, u^k_N, h^k_1, h^k_2, \ldots, h^k_N]^T \) is the state.

\( p_k \) is the control inputs.

\( u^k_i \) is the fluid velocity perturbations at cell \( i \) and time \( k\Delta t \).

\( h^k_i \) is the fluid depth perturbations at grid \( i \) and time \( k\Delta t \).

\( N \) is the number of cells used for the discretization.

**Kalman Filter:**

The Kalman filter is a technique that commonly used to estimate the state of a linear state space model. This linear state space model can be disturbed by some noise, mostly assumed as white noise. To improve the estimated state the Kalman filter uses measurement data that are related to the state (Simon, D., 2006). The Kalman filter consists of two steps:

1. The prediction step
2. The update step

In the first step the state is predicted with the linear state space model.

In the second step, the prediction is corrected with measurement data, so that the error covariance of the estimator is minimized. In this sense it is an optimal estimator.

This procedure is repeated for each time step with the state of the previous time step as initial value. Therefore the Kalman filter is called a recursive filter. The procedure of Kalman filtering technique is shown in Fig. 2.
The Kalman filtering technique is considered from the discrete time dynamical model and measurement data as follows

\[ x_{k+1} = Ax_k + Bp_k + w_k, \]  

and

\[ y_k = Cx_k + z_k. \]

where \( x_k \) is the state vector at time \( k \). 
\( A \) is the model operator. 
\( B \) is the input operator. 
\( p_k \) is the optimal input at time \( k \). 
\( w_k \) is the random noise affecting the model at time \( k \) with zero-mean and covariance matrix \( Q \). 
\( y_k \) is the measurement vector. 
\( C \) is the measurement operator. 
\( z_k \) is the measurement noise vector at time \( k \) with zero-mean and covariance matrix \( R \). 

The objective of the Kalman filtering technique is to obtain estimates \( \hat{x}_k \) of the state \( x_k \) using measurement \( y_k \). Therefore, the formulation of the Kalman filtering technique can be written as follows (Durran, D.R., 1999)

The prediction step:

\[ \hat{x}_{k+1} = A\hat{x}_k + Bp_k + \hat{w}_k, \]  

\[ \hat{P}_{k+1} = A\hat{P}_k A^T + Q. \]

The update step:

\[ K_k = \hat{P}_{k+1} C^T (C\hat{P}_{k+1} C^T + R)^{-1}, \]

\[ \hat{x}_{k+1} = \hat{x}_{k+1} + K_k (y_k - C\hat{x}_{k+1}), \]
\[ \hat{P}_{k+1} = (I - K_k C) \hat{P}_{k+1} \]  

(17)

In the above equations the “super minus” to be a priori state estimate at step \( k \) and a \( T \) superscript indicates matrix transposition. The \( K \) matrix is called the Kalman gain, and the \( P \) matrix is called the estimation error covariance.

The state estimate \( \hat{x} \) equation is fairly intuitive. The first term used to derive the state estimate at time \( k+1 \) is just \( A \) times the state estimate at time \( k \), plus \( B \) times the known input at time \( k \) from (13). The second term in the \( \hat{x} \) equation is called the correction term which represents the amount to correct the propagated state estimate due to measurement.

Investigation of the \( K \) equation shows that if the measurement noise is large, \( R \) will be large, so \( K \) will be small and the measurement \( y \) will not be given much credibility when computing the next \( \hat{x} \). On the other hand, if the measurement noise is small, \( R \) will be small, so \( K \) will be large and the measurement \( y \) will be given more credibility when computing the next \( \hat{x} \).

Numerical Experiment:

Modeling the uncertainties is done by adding a noise term \( w_k \) to the linear state space model (10) and the system noise is assumed to be white Gaussian noise. The flow diagram of the procedure is shown in Fig. 3.

![Flow diagram of the implementation procedure.](image)

**Fig. 3:** Flow diagram of the implementation procedure.

The initial conditions are \( u(x, 0) = 0 \) and \( h(x, 0) \). Let the spatial domain be periodic on the interval \( 0 \leq x \leq 100 \) km. Use \( \Delta x = 5 \) m and \( \Delta t = 0.1 \) s. The measured data for the fluid is generated by the exact solution (Vallis, G.K., 2006), as shown in Fig. 4 and Fig. 5. The random noise has a standard deviation of 0.05 m. The diagonal values of covariance matrix \( Q \) is \( 1 \times 10^{-3} \).
Fig. 4: Measurement data of the perturbation fluid velocity at the 4th grid.

Fig. 5: Measurement data of the perturbation fluid depth at the 4th grid.

Fig. 6 and Fig. 7 show the comparisons of states from three approaches; (i) exact solution (Exact) (ii) estimated state of linear state space shallow water model that uses the Kalman filtering technique by incorporating the measured data (Estimated) and (iii) finite difference method (FDM).

From these comparisons it can be seen that the Kalman filtering technique can estimate the states (perturbation fluid velocity and perturbation fluid depth) that are close to the exact solutions.

Conclusion:
In this paper, a technique to estimate the state of linear state space model for a fluid wave with no mean velocity is presented. The Kalman filter based on linearized one-dimensional shallow water equations is used to estimate the present state and to predict the future fluid state. The results of experiments show that Kalman filtering technique can estimate the fluid wave states and filtering out noise included in the measured data and the finite difference approximation. In future work, The Kalman filter will be applied to a two dimensional shallow water model of the atmosphere.

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Fig. 6: Comparison of the perturbation fluid velocity at the 4th grid.

Fig. 7: Comparison of the perturbation fluid depth at the 4th grid.

REFERENCES


