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Application of Reduced Differential Transform Method to the Wu-Zhang Equation

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Abstract: In this paper, the reduced differential transform method (RDTM) is presented for solving the Wu-Zhang equation, a (2+1)-dimensional nonlinear dispersive wave equation. In this method, the solution is calculated in the form of convergent power series with easily computable components. The results show that the proposed technique, without linearization or small perturbation, is very effective and convenient.

Key words: Reduced differential transform method; Wu-Zhang equation.

INTRODUCTION

The nonlinear partial differential equations (NPDEs) are encountered in various disciplines, such as physics, mechanics, chemistry, biology, mathematics and engineering. Many efforts have been made on the study of NPDEs. Long wave in shallow water is a subject of broad interests and has a long colorful history. Wu and Zhang derived three sets of model equations for modeling nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth [Wu and Zhang (1996)]. One of these equations, Wu-Zhang equation (which describes (2+1)-dimensional dispersive long wave), can be written as [Chen, Tang and Lou, Wu and Zhang (1996)].

$$\begin{aligned} &u_t + uu_x + vu_y + w_x = 0,\\ &v_t + uv_x + vv_y + w_y = 0,\\ &w_t + (uw)_x + (vw)_y + \tfrac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) = 0, \end{aligned} \tag{1}$$

where w is the elevation of the water, u is the surface velocity of water along x-direction, and v is the surface velocity of water along y-direction.

The explicit solutions of Eq. (1) are very helpful for costal and civil engineers to apply the nonlinear water wave model in harbor and coastal design. Recently introduced some new methods and are applied to nonlinear equations such as Variation Iteration Method [He (2000), Ganji, Tari and Babazadeh (2007)], Homotopy Perturbation Method [He (2000, 2005), Ganji (2006) and Ma (2008)], the homogeneous balance method[Fan and Zhang (1998)], the differential transform method (DTM) [Ayaz (2003), Kurnaz, Oturanc and Kiris (2005) and Bildik and Konuralp (2006)]. Among all of the methods, the reduced differential transform method (RDTM), employed for solving the nonlinear partial differential equations [Keskin and Oturanc (2010, 2010, 2010)]. The RDTM is presented to overcome the demerit of complex calculation of DTM. This technique doesn't require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computation.

In this paper, we apply the reduced differential transform method to find the approximation solution for the Wu-Zhang equation.

2 Reduced Differential Transform Method:

The basic definitions of reduced differential transform method are introduced as follows:

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Definition 2.1:

If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \tag{2}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

Definition 2.2:

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{3}$$

Then combining equation (2) and (3) we write

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k.$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion. The fundamental operations of reduced differential transform method are listed in Table 1 below.

Table 1: Reduced differential transformation

Original function	Transformed function
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x) \quad (\alpha\ is\ a\ constant)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m\delta(k-n), \quad \delta(k) = \begin{cases}1, k=0\\0. k\neq 0\end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)\dots(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$

3 Application of the RDTM to the Wu-Zhang Equation:

We consider the Wu-Zhang equation with the initial conditions:

$$u(x,y,0) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y), \tag{4}$$

$$v(x,y,0) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y), \tag{5}$$

$$w(x,y,0) = \frac{2}{3}\left(k_1{}^2 + k_2{}^2\right) sech^2(k_1 x + k_2 y), \tag{6}$$

where b_0, k_1, k_2 and k_3 are arbitrary constants.

For the solution procedure, we first take the differential transform of (2) by the use of Table 1 and have the following equations

$$\begin{aligned}(k+1)U_{k+1}(x,y) &= -\frac{\partial}{\partial x} W_k(x,y) - \sum_{r=0}^{k} U_r(x,y)\frac{\partial}{\partial x} U_{k-r}(x,y) \\ &\quad - \sum_{r=0}^{k} V_r(x,y)\frac{\partial}{\partial y} U_{k-r}(x,y),\end{aligned} \tag{7}$$

$$\begin{aligned}(k+1)V_{k+1}(x,y) &= -\frac{\partial}{\partial y} W_k(x,y) - \sum_{r=0}^{k} U_r(x,y)\frac{\partial}{\partial x} V_{k-r}(x,y) \\ &\quad - \sum_{r=0}^{k} V_r(x,y)\frac{\partial}{\partial y} V_{k-r}(x,y)\end{aligned} \tag{8}$$

$$\begin{aligned}(k+1)W_{k+1}(x,y) &= -\frac{1}{3}\left[\frac{\partial^3}{\partial x^3} U_k(x,y) + \frac{\partial^3}{\partial x \partial y^2} U_k(x,y) + \frac{\partial^3}{\partial x^2 \partial y} V_k(x,y) + \frac{\partial^3}{\partial y^3} V_k(x,y)\right] \\ &\quad - \frac{\partial}{\partial x}\left[\sum_{r=0}^{k} U_r(x,y) W_{k-r}(x,y)\right] \\ &\quad - \frac{\partial}{\partial y}\left[\sum_{r=0}^{k} V_r(x,y) W_{k-r}(x,y)\right],\end{aligned} \tag{9}$$

where the *t*-dimensional spectrum functions $U_k(x, y)$, $V_k(x, y)$ and $W_k(x, y)$ are the transformed functions. From the initial conditions (4-6), we will have

$$U_0(x,y) = u(x,y,0) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y), \tag{10}$$

$$V_0(x,y) = v(x,y,0) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y), \tag{11}$$

$$W_0(x,y) = w(x,y,0) = \frac{2}{3}\left({k_1}^2 + {k_2}^2\right) sech^2(k_1 x + k_2 y). \tag{12}$$

Now, substituting the initial conditions (10-12) into Eqs (7-9), we obtain the following $U_k(x, y)$, $V_k(x, y)$ and $W_k(x, y)$ values successively

$$U_1(x,y) = \frac{2\sqrt{3}}{3} k_1 k_3 sech^2(k_1 x + k_2 y)$$

$$U_2(x,y) = -\frac{2\sqrt{3}}{3} k_1 {k_3}^2 \tanh(k_1 x + k_2 y)\, sech^2(k_1 x + k_2 y)$$

$$U_3(x,y) = \frac{2\sqrt{3}}{9} k_1 {k_3}^3 (3\tanh^2(k_1 x + k_2 y) - 1) sech^2(k_1 x + k_2 y)$$

$$\vdots$$

$$V_1(x,y) = \frac{2\sqrt{3}}{3} k_2 k_3 sech^2(k_1 x + k_2 y)$$

$$V_2(x,y) = -\frac{2\sqrt{3}}{3} k_2 {k_3}^2 \tanh(k_1 x + k_2 y)\, sech^2(k_1 x + k_2 y)$$

$$V_3(x,y) = \frac{2\sqrt{3}}{9} k_2 {k_3}^3 (3\tanh^2(k_1 x + k_2 y) - 1) sech^2(k_1 x + k_2 y)$$

$$\vdots$$

and

$$W_1(x,y) = -\frac{4}{3}\left({k_1}^2 + {k_2}^2\right) k_3 \tanh(k_1 x + k_2 y)\, sech^2(k_1 x + k_2 y)$$

$$W_2(x,y) = \frac{2}{3}\left({k_1}^2 + {k_2}^2\right) {k_3}^2 (3\tanh^2(k_1 x + k_2 y) - 1) sech^2(k_1 x + k_2 y)$$

$$W_3(x,y) = -\frac{8}{9}\left({k_1}^2 + {k_2}^2\right) {k_3}^3 \tanh(k_1 x + k_2 y)\,(3\tanh^2(k_1 x + k_2 y) - 2) sech^2(k_1 x + k_2 y)$$

$$\vdots$$

Then, the inverse transformation of the set of values $\{U_k(x,y)\}_{k=0}^{3}$, $\{V_k(x,y)\}_{k=0}^{3}$ and $\{W_k(x,y)\}_{k=0}^{3}$ gives four term (Order 3) approximation solution as

$$\begin{aligned}\tilde{u}_3(x,y,t) = \sum_{k=0}^{3} U_k(x,y)t^k &= -\frac{k_3+k_2b_0}{k_1} + \frac{2\sqrt{3}}{3}k_1\tanh(k_1x+k_2y)\\ &+\frac{2\sqrt{3}}{3}k_1k_3 sech^2(k_1x+k_2y)t\\ &-\frac{2\sqrt{3}}{3}k_1{k_3}^2\tanh(k_1x+k_2y)\, sech^2(k_1x+k_2y)t^2\\ &+\frac{2\sqrt{3}}{9}k_1{k_3}^3(3\tanh^2(k_1x+k_2y)-1)sech^2(k_1x+k_2y)t^3,\end{aligned}$$

$$\begin{aligned}\tilde{v}_3(x,y,t) = \sum_{k=0}^{3} V_k(x,y)t^k &= b_0 + \frac{2\sqrt{3}}{3}k_2\tanh(k_1x+k_2y) + \frac{2\sqrt{3}}{3}k_2k_3 sech^2(k_1x+k_2y)t\\ &-\frac{2\sqrt{3}}{3}k_2{k_3}^2\tanh(k_1x+k_2y)\, sech^2(k_1x+k_2y)t^2\\ &+\frac{2\sqrt{3}}{9}k_2{k_3}^3(3\tanh^2(k_1x+k_2y)-1)sech^2(k_1x+k_2y)t^3,\end{aligned}$$

$$\begin{aligned}\tilde{w}_3(x,y,t) = \sum_{k=0}^{3} W_k(x,y)t^k &= \frac{2}{3}({k_1}^2+{k_2}^2)sech^2(k_1x+k_2y)\\ &-\frac{4}{3}({k_1}^2+{k_2}^2)k_3\tanh(k_1x+k_2y)\, sech^2(k_1x+k_2y)t\\ &+\frac{2}{3}({k_1}^2+{k_2}^2){k_3}^2(3\tanh^2(k_1x+k_2y)-1)sech^2(k_1x+k_2y)t^2\\ &-\frac{8}{9}({k_1}^2+{k_2}^2){k_3}^3\tanh(k_1x+k_2y)\,(3\tanh^2(k_1x+k_2y)-2)sech^2(k_1x\\ &+k_2y)t^3.\end{aligned}$$

Therefore, the exact solution of Wu-Zhang equation is given by

$$u(x,y,t) = \lim_{n\to\infty}\tilde{u}_n(x,y,t),$$

$$v(x,y,t) = \lim_{n\to\infty}\tilde{v}_n(x,y,t),$$

$$w(x,y,t) = \lim_{n\to\infty}\tilde{w}_n(x,y,t).$$

This solution is convergent to the exact solution

$$u(x,y,t) = -\frac{k_3+k_2b_0}{k_1} + \frac{2\sqrt{3}}{3}k_1\tanh(k_1x+k_2y+k_3t),$$

$$v(x,y,t) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y + k_3 t),$$

$$w(x,y,t) = \frac{2}{3}\left({k_1}^2 + {k_2}^2\right) sech^2 (k_1 x + k_2 y + k_3 t),$$

and the same as approximate solution of the homotopy perturbation method [Ma (2008)]. (see figures 1,2 and 3)

In figures 1, 2 and 3 the final results obtained from RDTM were compared with the results of the exact solution when b_0 = 0.1, k_1 = 0.05, k_2 = 0.1, k_3 = 0.3 and t = 0.2. The comparison shows a good agreement between the results.

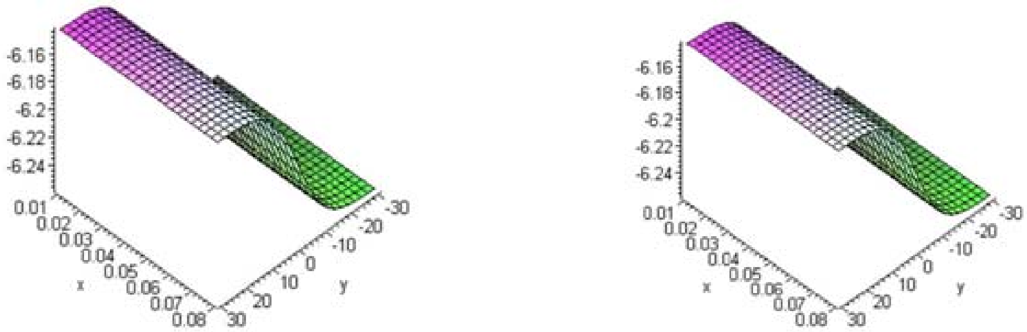

Fig. 1: Comparison between the exact solution and the 3th-order RDTM of *u(x, y, t)*.

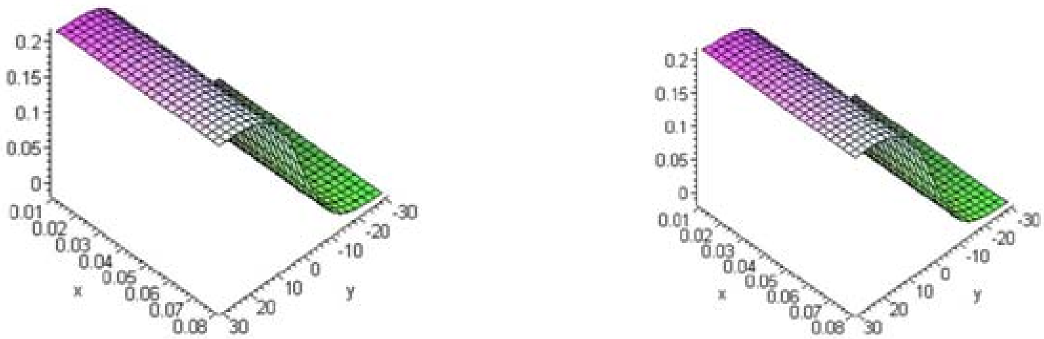

Fig. 2: Comparison between the exact solution and the 3th-order RDTM of *v(x, y, t)*.

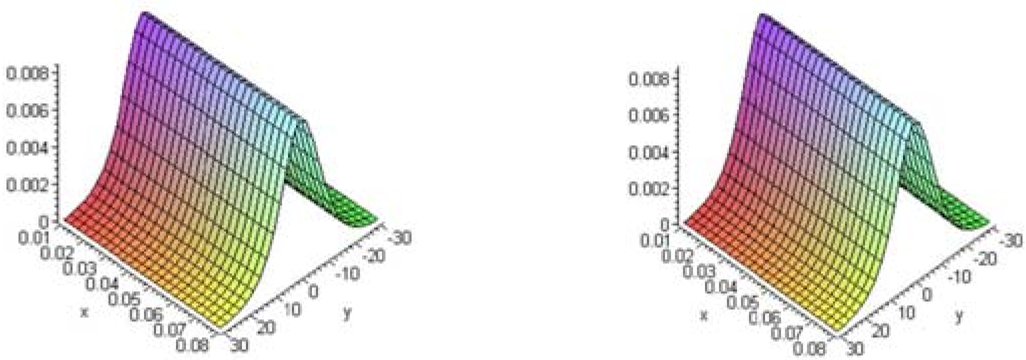

Fig. 3: Comparison between the exact solution and the 3th-order RDTM of *w(x, y, t)*.

4 Conclusion:

In this paper, the reduced differential transform method (RDTM) was used for finding the approximate solution of Wu-Zhang equation. The obtained solution are compared with the exact solution and the HPM. The results reveal that the RDTM, with less and easier computations, has the same results, as HPM.

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