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Exact Solution of Burgers Equations by Homotopy Perturbation Method and Reduced Differential Transformation Method

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Abstract: In this paper, we will compare the Reduced differential transform method (RDTM) and Homotopy Perturbation Method (HPM) for solving Burgers equation. The analytical results of the equations have been obtained in terms of convergent series with easily computable components. The results obtained by the proposed method (RDTM) are compared with the results obtained by (HPM). Consequently, the results of our system tell us the two methods can be alternative ways for solution of the linear and nonlinear higher-order initial value problems.

Key words: Burgers equation; Homotopy perturbation method; Time-dependent partial differential equations; Reduced Differential Transformation Method.

INTRODUCTION

As the researches indicated, the nonlinear equations are one of the most important phenomena across the world. Nonlinear phenomena have important effects on applied mathematics, physics, and issues related to engineering. Then the variation of each parameter depends on different factors. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations (NLPDEs) in physics and mathematics is still a big problem that needs new methods to discover new exact or approximate solutions. Most of nonlinear equations do not have a precise analytic solution. There are also some analytic techniques for nonlinear equations. Some of the classic analytic methods are perturbation techniques, and d-expansion method. In the recent years, many authors mainly had paid attention to study solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM), tanh method, homotopy perturbation method (HPM) (He, 1999; He, 2000; He, 2005; He, 2003; He, 2004; He, 1997) sinh-cosh method, HAM, the DTM, and variational iteration method (VIM). Reduced Differential Transformation Method (RTDM) (Zhou, 1986; Arikoglu, 2005; Ayaz, 2004; Liu, 2007; Ertrk, 2007; Hassan, 2008). In this paper, we employ Reduced. Differential Transform Method (RTDM), Homotopy Perturbation Method (HPM) to solve Burgers equations in position (2+1)-dimensional, (3+1)-dimensional and (n +1)-dimensional with boundary conditions. The concept of differential transform was first introduced by Zhou, it is a semi-numerical-analytic-technique that formalizes Taylor series in a totally different manner. With this technique, the given differential equation and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful to obtain exact and approximate solutions of linear and nonlinear differential equations.

The concept of the homotopy perturbation method (HPM) was first proposed by He, which is developed by combining the standard homotopy and perturbation method. In these method the solution is given in an infinite series usually convergent to an accurate solution. The (HPM) does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter. This paper is organized as follows: In Section 2 the Homotopy perturbation method is described. In Section 3 Solution of Burgers equation by (HPM). In Section 4, the Reduced differential transformation method is described. In Section 5 Solution of Burgers equation by (RTDM), and conclusion is given in Section 6.

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2. *Basic Ideas of Homotopy Perturbation Method:*
Fundamentals:

To illustrate the basic ideas of this method, we consider the following general nonlinear differential equation

$$A(r) - f(r) = 0 r \in \Omega \tag{1}$$

Considering the boundary condition of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0 \quad r \in \Gamma$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain . The operator A can, generally speaking, be divided in to two parts L and N, where L is linear, and N is nonlinear, therefore Eq. (1) can be written as

$$L(u) + N(u) - f(r) = 0. \qquad r \in \Omega$$

By using homotopy technique, one can construct a homotopy

$$v(r,p): \Omega \times [0,1] \to R$$

which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{2}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\big(N(v) - f(r)\big) = 0$$

where p ∊ [0,1] is an embedding parameter, and u_0 is the initial approximation of Eq. (1) which satisfies the boundary conditions. Clearly, we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{3}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{4}$$

The changing process of p from zero to unity is just that of $v(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If, the embedding parameter p; $(0 \le p \le 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can naturally assume that the solution of Eqs.
(3) and (4) can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \cdots$$

and setting $p = 1$ results in the approximate solution of Eq. (2) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots$$

The combination of the perturbation method and the homotopy method is called the HPM. The series (5) is convergent for most cases. However, the convergence rate depends on the nonlinear operator *A (v)*.

The second derivative of *N (v)* with respect to *v* must be small because the parameter *p* may be relatively large, i.e. p → 1. The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

3. *Application HPM for Burgers Equation:*

3.1. *(2+1) -Dimensional Burgers Equation:*

We will consider (2+1)-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \alpha\left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y}\right) - \beta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{6}$$

With initial condition

$$u(x, y, 0) = x + y \tag{7}$$

We first construct homotopy as follows

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + \alpha\left(v\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \beta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right) = 0. \tag{8}$$

Assume the solution of Eq. (8) in the form

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{9}$$

Substitution (9) into (8), and comparing coefficients of the terms with the identical powers of *p*, lead to

$$\begin{cases} p^0 : \dfrac{\partial v_0}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0 \\ p^1 : \dfrac{\partial v_1}{\partial t} + \dfrac{\partial u_0}{\partial t} + \alpha v_0 \dfrac{\partial v_0}{\partial x} + \alpha v_0 \dfrac{\partial v_0}{\partial y} - \beta \dfrac{\partial^2 v_0}{\partial x^2} = 0 \end{cases}$$

$$\begin{cases} p^j : \dfrac{\partial v_j}{\partial t} + \alpha\left(\displaystyle\sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} + v_k \frac{\partial v_{j-k-1}}{\partial y}\right) - \beta\left(\dfrac{\partial^2 v_{j-1}}{\partial x^2} + \dfrac{\partial^2 v_{j-1}}{\partial y^2}\right) = 0. \\ \qquad \forall j \geq 2. \end{cases}$$

Now try to obtain a solution for equation (8) with initial condition (7), subsequently solving the above equations we have

$$v_0 = x + y$$

$$v_1 = -2\alpha t(x + y)$$

$\vdots$
$v_n = (-2\alpha t)^n(x+y).$

We will obtain

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$$

$$= (x+y) + (-2\alpha t)(x+y) + (-2\alpha t)^2(x+y) + (-2\alpha t)^3(x+y) + \cdots$$

$$= (x+y) + (1 - 2\alpha t + (-2\alpha t)^2 + (-2\alpha t)^3 + \cdots)$$

$$= \frac{x+y}{1+2\alpha t}.$$

Which is an exact solution of equations system (6).

3.2. ***(3+1)-dimensional Burgers Equation:***

We will consider (3+1)- dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \alpha\left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial z}\right) - \beta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0. \tag{10}$$

With initial condition

$$u(x,y,z,0) = x + y + z\,. \tag{11}$$

We first construct homotopy as follows

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p(\frac{\partial v}{\partial t} + \alpha\left(v\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial z}\right) - \beta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)) = 0. \tag{12}$$

Assume the solution of Eq. (12) in the form

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{13}$$

Substitution (13) into (12), and comparing coefficients of the terms with the identical power of p, lead to

$$\begin{cases} p^0 : \dfrac{\partial v_0}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0 \\ p^1 : \dfrac{\partial v_1}{\partial t} + \dfrac{\partial u_0}{\partial t} + \alpha v_0\dfrac{\partial v_0}{\partial x} + \alpha v_0\dfrac{\partial v_0}{\partial y} + \alpha v_0\dfrac{\partial v_0}{\partial z} - \beta\dfrac{\partial^2 v_0}{\partial x^2} - \beta\dfrac{\partial^2 v_0}{\partial y^2} - \beta\dfrac{\partial^2 v_0}{\partial z^2} = 0 \end{cases}$$

$$\begin{cases} p^j : \dfrac{\partial v_j}{\partial t} + \alpha(\displaystyle\sum_{k=0}^{j-1} v_k\frac{\partial v_{j-k-1}}{\partial x} + v_k\frac{\partial v_{j-k-1}}{\partial y} + v_k\frac{\partial v_{j-k-1}}{\partial z}) - \beta\left(\frac{\partial^2 v_{j-1}}{\partial x^2} + \frac{\partial^2 v_{j-1}}{\partial y^2} + \frac{\partial^2 v_{j-1}}{\partial z^2}\right) = 0. \\ \forall j \ge 2. \end{cases}$$

Now try to obtain a solution for equation (12) with initial condition (11) subsequently solving The above equations we have

$$v_0 = x + y + z$$

$$v_1 = -3\alpha t(x + y + z)$$

$$\vdots$$

We will obtain

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$$

$$= (x + y + z) + (-3\alpha t)(x + y + z) + (-3\alpha t)^2(x + y + z) + \cdots$$

$$= (x + y + z)(1 - 3\alpha t + (-3\alpha t)^2 + (-3\alpha t)^3 + \cdots \quad = \frac{(x + y + z)}{1 + 3\alpha t}.$$

Which is an exact solution of equations system (10).

3.3. (N +1)-*dimensional Burgers Equation:*
We will consider (n + 1)- dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \alpha\left(u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} + \cdots + u\frac{\partial u}{\partial x_n}\right) - \beta\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}\right) = 0. \tag{14}$$

With initial condition

$$u(x_1, x_2, \cdots, x_n, 0) = x_1 + x_2 + \cdots + x_n. \tag{15}$$

We first construct homotopy as follows

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) p\left(\frac{\partial v}{\partial t} + \alpha\left(v\frac{\partial v}{\partial x_1} + v\frac{\partial v}{\partial x_2} + \cdots + v\frac{\partial v}{\partial x_n}\right) - \beta\left(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \cdots + \frac{\partial^2 v}{\partial x_n^2}\right)\right) = 0. \tag{16}$$

(16) Assume the solution of Eq. (16) in the form

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{17}$$

Substitution (17) into (16), and comparing coefficients of the terms with the identical powers of *p*, lead to

$$\begin{cases} p^0 : \dfrac{\partial v_0}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0 \\ p^1 : \dfrac{\partial v_1}{\partial t} + \dfrac{\partial u_0}{\partial t} + \alpha v_0\dfrac{\partial v_0}{\partial x_1} + \alpha v_0\dfrac{\partial v_0}{\partial x_2} + \cdots + \alpha v_0\dfrac{\partial v_0}{\partial x_n} - \beta\dfrac{\partial^2 v_0}{\partial x_1^2} - \beta\dfrac{\partial^2 v_0}{\partial x_2^2} - \cdots - \beta\dfrac{\partial^2 v_0}{\partial x_n^2} = 0. \end{cases}$$

$$\begin{cases} p^j : \frac{\partial v_j}{\partial t} + \alpha(\sum_{k=0}^{j-1}\sum_{i=1}^{n} v_k \frac{\partial v_{j-k-1}}{\partial x_i}) - \beta(\frac{\partial^2 v_{j-1}}{\partial x^2}) = 0. \\ \qquad\qquad \forall j \geq 2. \end{cases}$$

Now try to obtain a solution for equation (16) with initial condition (15), subsequently solving the above equations we have

$$v_0 = x_1 + x_2 + \cdots + x_n$$

$$v_1 = -n\alpha t(x_1 + x_2 + \cdots + x_n)$$

$$\vdots$$
$$v_n = (-n\alpha t)^n(x_1 + x_2 + \cdots + x_n).$$

We will obtain

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots$$

$$= (x_1 + x_2 + \cdots + x_n) + (-n\alpha t)(x_1 + x_2 + \cdots + x_n) + (-n\alpha t)^2(x_1 + x_2 + \cdots + x_n) + \cdots$$

$$+(-n\alpha t)^n(x_1 + x_2 + \cdots + x_n) + \cdots$$

$$= (x_1 + x_2 + \cdots + x_n)(1 - n\alpha t + (-n\alpha t)^2 + (-n\alpha t)^3 + \cdots)$$

$$= \frac{(x_1 + x_2 + \cdots + x_{n)}}{1 + n\alpha t}.$$

Which is an exact solution of equations system (14)

4. *Basic Ideas of Reduced Differential Transformation Method:*

Fundamentals:The basic definitions of reduced differential transform method are introduce as follows

Definition 4.1:

If function *u (x,t)* is analytic and differentiated continuously with respect to time t and space *x* in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=t_0} \tag{18}$$

where the t-dimensional spectrum function U_k *(x)* is the transformed function. and the inverse transformation is defined as

$$u(x,t) = \sum_{k=1}^{\infty} U_k(x)(t - t_0)^k. \tag{19}$$

Then combining equation (18) and (19) we write

$$u(x,t) = \sum_{k=1}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right](t-t_0)^k. \tag{20}$$

Eq. (20) implies that the concept of reduced differential transformation is derived from the Taylor series expansion.

The following theorems that can be deduced from Eqs. (18) and (19) are given below (Arikoglu, 2006; Ertrk, 2007).

Theorem 1:

If *u (x) = y (x) ± z (x), then U (k) = Y (k) ± Z (k).*

Theorem 2:

If *f (x) = cg (x), then F (k) = cG (k), where is a constant.*

Theorem 3:

If $f(x) = \frac{d^n g(x)}{dx^n}$, then $F(k) = \frac{(k+n)!}{k!} G(k+n)$.

Theorem 4:

If $f(x) = g(x)h(x)$, then $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 5:

If $u(x) = x^n$ then . $U(k) = \delta(k-n), \delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$

5. Application RTDM for Burgers Equation:

5.1. (2 + 1)- ***Dimensional Burgers equation:***

We will consider (2+1)- dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \alpha\left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y}\right) - \beta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

With initial condition

$U(x,y,0) = x + y.$

According to the RDTM, we can construct the following iteration formula

$$(K+1)U_{k+1} = -\alpha\left[\sum_{j=0}^{k} U(j)\frac{\partial}{\partial x}U(k-j) + \sum_{j=0}^{k} U(j)\frac{\partial}{\partial y}U(k-j)\right] - \beta[\frac{\partial^2}{\partial x^2}U(k) + \frac{\partial^2}{\partial y^2}U(k)]$$

Now, we obtain the following U_k values successively

$$U_1 = -2\alpha(x+y), \qquad U_2 = (-2\alpha)^2(x+y), U_3 = (-2\alpha)^3(x+y), \cdots, U_n = (-2\alpha)^n(x+y).$$

Finally the differential inverse transform of U_k gives

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k t^k$$
$$= (x+y) + (-2\alpha t)(x+y) + (-2\alpha t)^2(x+y) + \cdots + (-2\alpha t)^n(x+y) + \cdots$$

$$= \frac{x+y}{1+2\alpha t}.$$

Which is an exact solution of equations system (6)

5.2. (3 + 1)-*dimensional Burgers Equation:*

We will consider (3 + 1)dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \alpha\left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial z}\right) - \beta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0.$$

With initial condition

$$U(x,y,z,0) = x + y + z$$

According to the RDTM, we can construct the following iteration formula

$$(k+1)U_{k+1} = -\alpha[\sum_{j=0}^{k} U(j)\frac{\partial}{\partial x}U(k-j) + \sum_{j=0}^{k} U(j)\frac{\partial}{\partial y}U(k-j) + \sum_{j=0}^{k} U(j)\frac{\partial}{\partial z}U(k-j)]$$

$$-\beta[\left[\frac{\partial^2}{\partial x^2}U(k) + \frac{\partial^2}{\partial y^2}U(k) + \frac{\partial^2}{\partial z^2}U(k)\right].$$

Now, we obtain the following U_k values successively

$$U_1 = -3\alpha(x+y+z), U_2 = (-3\alpha)^2(x+y+z), U_3 = (-3\alpha)^3(x+y+z),$$

$$\cdots, U_n = (-3\alpha)^n(x+y+z) \; .$$

Finally the differential inverse transform of U_k gives

$$u(x,y,z,t) = \sum_{k=0}^{\infty} U_k t^k$$

$$= (x+y+z) + (-3\alpha t)(x+y+z) + (-3\alpha t)^2(x+y+z) + \cdots + (-3\alpha t)^n(x+y+z) + \cdots$$

$$= \frac{x+y+z}{1+3\alpha t}.$$

Which is an exact solution of equations system (10).

5.3. (*n* + 1)-*Dimensional Burgers Equation:*

We will consider $(n+1)$-dimensional Burgers equation

$$\frac{\partial u}{\partial t}+\alpha\left(u\frac{\partial u}{\partial x_1}+u\frac{\partial u}{\partial x_2}+\cdots+u\frac{\partial u}{\partial x_n}\right)-\beta\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}\right)=0.$$

With initial condition

$$U(x_1,x_2,\cdots,x_n,0)=x_1+x_2+\cdots+x_n.$$

According to the RDTM, we can construct the following iteration formula

$$(k+1)U_{k+1}=-\alpha[\sum_{j=0}^{k}U(j)\frac{\partial}{\partial x_1}U(k-j)+\sum_{j=0}^{k}U(j)\frac{\partial}{\partial x_2}U(k-j)+\cdots\sum_{j=0}^{k}U(j)\frac{\partial}{\partial x_n}U(k-j)]$$

$$-\beta\left[\frac{\partial^2}{\partial x_1^2}U(k)+\frac{\partial^2}{\partial x_2^2}U(k)+\cdots+\frac{\partial^2}{\partial x_n^2}U(k)\right].$$

Now, we obtain the following U_k values successively

$$U_1=-n\alpha(x_1+x_2+\cdots+x_n)$$

$$U_2=(-n\alpha)^2(x_1+x_2+\cdots+x_n)$$

$$\vdots$$
$$U_n=(-n\alpha)^n(x_1+x_2+\cdots+x_n)$$

Finally the differential inverse transform of U_k gives

$$u(x_1,x_2,\cdots,x_n,t)=\sum_{k=0}^{\infty}U_k t^k$$

$$=(x_1+x_2+\cdots+x_n)+(-n\alpha t)(x_1+x_2+\cdots+x_n)+(-n\alpha t)^2(x_1+x_2+\cdots+x_n)+\cdots$$

$$+(-n\alpha t)^n(x_1+x_2+\cdots+x_n)+\cdots$$

$$=\frac{x_1+x_2+\cdots+x_n}{1+n\alpha t}.$$

Which is an exact solution of equations system (14).

Conclusion:

The main concern of this article is to conduct a comparative study between the Reduced Differential transformation method (RDTM) and Homotopy perturbation method (HPM). The comparison ofthe results between (RDTM), (HPM) and exact solution were shown a very interesting agreement which confirms the validity and high accuracy of the (RDTM). Too the results tell us the two successfully methods can be alternative way for the solution of the linear and nonlinear higherorder initial value problems. The two methods are so powerful and efficient that they both give approximations of higher accuracy and closed form solutions if existing.

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