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Vibration Analysis of a Variable Thickness Isotropic Kirchhoff Annular Plate Covered with Piezoelectric Layers Using Modified Wave Method

[1]N. Rasekh Saleh, [2]M.N. Bahrami, [3]M.H. KargarNovin

[1]Department of Mechanical and Aerospace Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran.
[2]Department of Mechanical Engineering, University of Tehran, Tehran, Iran.
[3]Department of Mechanical Engineering, Sharif University of Technology, Tehran, Iran.

Abstract: In this paper, based on Kirchhoff plate theory (CLPT), vibration analysis of an arbitrary nonlinear variable thickness annular isotropic plate which is covered with piezoelectric layers on the top and bottom surfaces in a short circuit state is performed using a semi-analytic method, named modified wave method (MWM). The plate is partitioned into several continuous segments with constant thicknesses, for which there exists exact analytical solutions. Considering wave method, the solution of each segment is assumed to be the combination of waves moving forward and backward, and at each step, waves in positive and negative directions are obtained in terms of waves at the inner edge of the first segment. Using continuity condition, the propagation, transmission and reflection matrices for each segment are extracted. Using these matrices the positive and negative waves at the last boundary condition are obtained in terms of waves of the first boundary condition. Satisfying the boundary conditions, a system of homogeneous linear equations is achieved. The nontrivial roots of the characteristic equation of these simultaneous linear equations are the natural frequencies of the plate. To verify the modified wave method presented here, the results are compared to the results of analytical and FE, which have proven to be of high accuracy and excellent agreement which validates the present approach. Therefore, this method can also be used to calculate the natural frequencies of plates with any arbitrary variable thickness.

Key words: vibration; variable thickness annular plate; Piezoelectric layers; short circuit; CLPT; semi-analytical, modified wave method; Transfer, propagation and transmission Matrices.

INTRODUCTION

Circular and annular geometries are used in a wide variety of applications and so extensive research has been done on them. Leissa, extensively worked on Vibration of plates 0, 0, 0, 0, 0, 0, (Leissa, 1981). The vibration analysis of piezoelectric beams and plates has been of special importance, especially for sound generation purposes in ultra-sonic engines, for instance: Bailey et al, studied distributed piezoelectric-polymer active vibration of a cantilever beam 0(Bailey *et al., 1985*), Crawley studied piezoelectric actuators in intelligent structures(Crawley *et al., 1987*), Heyliger worked on Free vibration of laminated piezoelectric plates and discs(Heyliger *et al.,* 1999), L. Wang et al, studied the design of smart functionally graded thermo-piezoelectric composite structure(Wang *et al., 2001*), Q Wang et al, studied analysis of piezoelectric coupled circular plate(Wang *et al., 2001*). Duan, Quek and Wang, studied vastly free vibration of piezoelectric coupled thin and thick annular plate 0(Duan *et al., 2005*), using Kirchhoff and Mindlin plate theories.

Some Researchers have studied wave propagations in elastic solids 0, 0(Doyle, 1989), 0(Graf), 0(Kolsky,1963), 0(Harland *et al., 2000*) and (Mei, 2002) presented a systematic technique for describing wave transmission and reflection in uniform one-dimensional waveguides. (Lee, 20070), (Nik-khah Bahrami, 2008) and (Loghmani, 2008) used a wave approach to analyze the non-uniform one-dimensional waveguides having a polynomial or exponential cross-section. Khoshbayani, Rasekh Saleh and Nik-khah Bahrami have developed modified wave propagation to analyze arbitrary variable section rods and beams (Khoshbayani *et al.,* 2011)0, which is in fact a generalization of(Achenbach, 1973; Doyle, 1989; Graf; Kolsky, 1963; Harland *et al.,* 2000; Mei, 2002; Lee *et al.,* 2007; Nikkhah-Bahrami *et al.,* 2008; Loghmani *et al.,*2008).

In this work, semi-analytical 'Modified Wave Method' for continuous elements is applied to analyze the vibration of an isotropic arbitrary variable thickness annular plate, which is covered with piezoelectric layers on the top and the bottom surfaces in a short circuit state, based on Kirchhoff plate theory. The method presented here is similar to the one we've presented in 0(Khoshbayani *et al.,* 2011), of course some vital changes are done first to adopt the solution to the MWM. To validate the approach applied in this paper, the results of this survey

Crossponding Author: N. Rasekh Saleh, Department of Mechanical and Aerospace Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran

are compared to those attained via analytical and finite element analysis, which have shown excellent agreements.

Nomenclature

Symbol	Definition	Symbol	Definition	Symbol	Definition
Subscripts 'p'	Piezo	Z_{i1}	ordinary and modified Bessel functions of the first kind	M_{rr} & $M_{r\theta}$	Moments
Subscripts 'a' & 'b'	Cross Section 'a' & 'b'	Z_{i2}	ordinary and modified Bessel functions of the second kind	q_r	Shearing force
Subscripts $'r'$, $'\theta'$,' z'	Cylindrical coordinates	$J_m(\delta_i r)$	Bessel functions of the first kind	V_r	effective Shearing force
Subscripts 1,2 & 3	Cylindrical coordinates	$I_m(\delta_i r)$	Modified Bessel functions of the first kind	$\{z\}$	State vector at each point
E	Elasticity Modulus	$Y_m(\delta_i r)$	Bessel functions of the second kind	$[t_{ij}]$	a matrix, which relates state vector to moving waves
ρ	Density	$K_m(\delta_i r)$	Modified Bessel functions of the second kind	Δ	Laplacian operator in polar coordinate
v	Poisson's ratio	P_0, d_1, d_2 and A_1	Constants depend on geometrical and material properties	$[F^+]$ & $[F^-]$	propagation matrices
h	thickness	x_i	Roots of cubic equation	$[\psi^{\pm}]$ & $[\phi^{\pm}]$	so-called displacement and internal force matrices
σ_i	stress components	λ_i	A function of x_i	$[R]$ & $[T]$	reflection and transmission matrices
ε_k	strain components	δ_i	$\sqrt{\lvert\lambda_i\rvert}$	$\begin{bmatrix}\mu & \lambda \\ \eta & \beta\end{bmatrix}$	A matrix which relates waves at different steps together
C_{ki}	stiffness matrix components	δ_i r	Arguments of Bessel functions	hi	Host plate thickness at inner edge
E_k	Electric field components	ω	Angular natural frequency (rad/s)	h_o	Host plate thickness at outer edge
e_{ji}	permeability constant	f	Natural frequency (Hz)	p	Nonlinear power of the radius
D_i	electric displacement components	c_1 to c_{12}	Integration constants		
Ξ_{ij}	dielectric coefficients	d_7 *to* d_{12}	Arguments of exponential functions		
$\varphi(r,\theta,t)$	potential field	$\pm\delta_j.i \pm b_j$	Arguments of exponential functions		
u_r, u_θ, u_z	Radial, circumferential and lateral displacements	$\pm\delta_j \pm b_j$	Arguments of exponential functions		
$w(r,\theta,t)= u_z$	lateral displacement	a_j^+ & a_j^-	Moving waves		

Problem Definition:

Vibration analysis of a nonlinear variable thickness isotropic annular plate, which is covered with piezoelectric layers in a short circuit state, based on Kirchhoff plate theory, is the goal of this survey. The thickness of the plate is assumed to be an arbitrary ax-symmetric nonlinear function of radial coordinate (Fig.1).

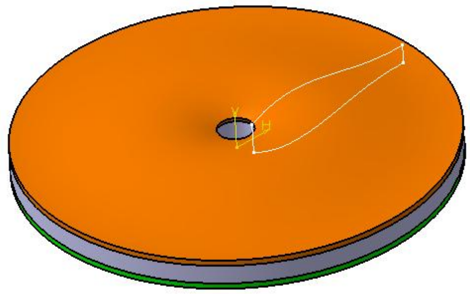

Fig.1 Variable Thickness Annular Plate, Covered With Piezoelectric Layers

For the piezoelectric layers, one can write the stress-strain-electrical relationship as follows 0(Wang *et al., 2001*):

$$\sigma_i = C_{ki}\varepsilon_k - e_{ji}E_k \qquad i, j, k = 1,2,3 \tag{1}$$

In which σ_i and ε_k represent the stress and strain tensor components respectively and e represents the permeability constant of piezoelectric material, and E_k indicates the components of the electric field. Also C_{ki} are components of symmetric stiffness matrix. Using Classical plate theory, the last equation for a piezo plate is expanded as:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix} = \begin{pmatrix}\bar{C}_{11}^E & \bar{C}_{12}^E & 0\\ \bar{C}_{12}^E & \bar{C}_{11}^E & 0\\ 0 & 0 & \bar{C}_{11}^E - \bar{C}_{12}^E\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{pmatrix} - \begin{pmatrix}0 & 0 & \bar{e}_{31}\\ 0 & 0 & \bar{e}_{31}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}E_1\\ E_2\\ E_3\end{pmatrix} \tag{2}$$

in which

$$\overline{C}_{11}^E = C_{11}^E - {(C_{13}^E)^2}/{C_{33}^E}\ ;\ \overline{C}_{12}^E = C_{12}^E - {(C_{13}^E)^2}/{C_{33}^E}\ \ and\ \ \overline{e}_{13}^E = e_{31}^E - {C_{11}^E e_{33}}/{C_{33}^E} \tag{3}$$

In addition, the electric displacement-strain relation for the symmetry piezoelectric material is given by:

$$\begin{pmatrix}D_1\\ D_2\\ D_3\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \bar{e}_{13} & \bar{e}_{31} & 0\end{pmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{Bmatrix} + \begin{pmatrix}\overline{\Xi}_{11} & 0 & 0\\ 0 & \overline{\Xi}_{11} & 0\\ 0 & 0 & \overline{\Xi}_{11}\end{pmatrix}\begin{Bmatrix}E_1\\ E_2\\ E_3\end{Bmatrix} \tag{4}$$

in which D_i (i=1,2,3) represent components of the electric displacement, $\overline{E}_{11}, \overline{E}_{33}$ are the symmetric reduced dielectric of piezoelectric layer and given by0 (Wang *et al., 2001*):

$$\overline{\Xi}_{33} = \Xi_{33} + ({e_{33}^2}/{C_{33}^E})\ ;\ \overline{\Xi}_{11} = \Xi_{11} \tag{5}$$

in which Ξ_{11}, Ξ_{33} are the dielectric coefficients of piezoelectric. Note that in polar coordinate 1, 2, 3 are respectively r, θ and z coordinates. The potential field in the piezo layer is assumed to be as (Duan *et al.,* 2005):

$$\phi(r,\theta,z,t) = \varphi(r,\theta,t)\sin\left(\pi\left(z - h_f\right)/h_p\right) \tag{6}$$

Applied Procedure: the Application of Modified Wave Method (MWM):

Using Modified Wave Method a typical plate is partitioned to several continuous segments with constant cross-sections (Fig.2). The solution to the vibration of each segment should be found first. Then waves at the entrance of an arbitrary continuous segment in positive and negative directions are propagated and transmitted to another segment, which could be expressed in terms of the waves at the initial segment. Then by satisfying the boundary conditions, a characteristic equation is obtained and the natural frequencies are calculated.

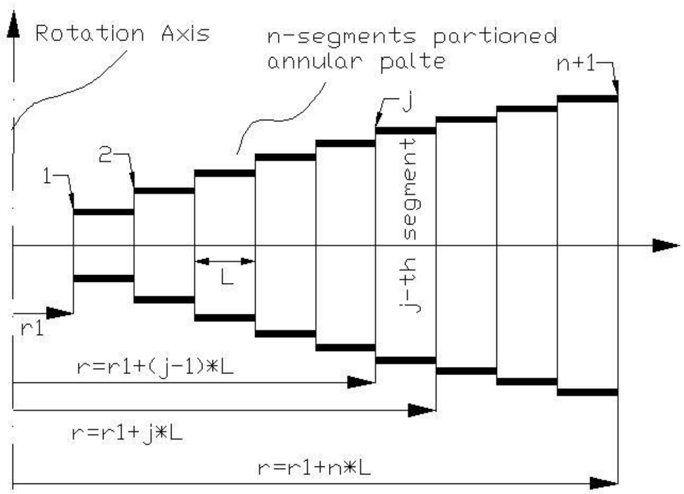

Fig.2 Partitioned Variable Thickness Annular Plate, Covered With Piezoelectric Layers

Analytical Vibration Analysis of a Constant Thickness Kirchhoff Annular Plate Covered with Piezoelectric Layers:

For the Kirchhoff plate covered with piezo layers, one should find $w(r,\theta,t)$ and $\varphi(r,\theta,t)$, to have all mechanical and potential fields obvious. According to the results of 0(Wang *et al., 2001*) and (Duan *et al.*, 2005), one can write the analytical solution to the vibration of a constant thickness isotropic Kirchhoff annular plate which is covered with piezoelectric layers, as follows:

$$w(r,\theta,t) = \sum_{i=1}^{3}[c_i Z_{i1}(m\,,\delta_i r) + c_{i+3} Z_{i2}(m,\delta_i r)]\cos(m\,\theta)\,e^{i\omega t} \tag{7}$$

$$\varphi(r,\theta,t) = \sum_{i=1}^{3} x_i[c_i Z_{i1}(m\,,\delta_i r) + c_{i+3} Z_{i2}(m,\delta_i r)]\cos(m\,\theta)\,e^{i\omega t} \tag{8}$$

In which, $\mathbf{c_i}$ are integration constants, given by boundary conditions and $\mathbf{Z_{i1}}$ and $\mathbf{Z_{i2}}$ are ordinary and modified Bessel functions of the first and second kind:

$$Z_{i1}(m\,,\delta_i r) = \begin{cases} J_m(\delta_i r), \lambda_i < 0 \\ I_m(\delta_i r), \lambda_i > 0 \end{cases} \qquad Z_{i2}(m\,,\delta_i r) = \begin{cases} Y_m(\delta_i r), \lambda_i < 0 \\ K_m(\delta_i r), \lambda_i > 0 \end{cases} \tag{9}$$

Where

$$\delta_i = \sqrt{|\lambda_i|} = \sqrt{\left|\frac{2\pi^2 \overline{\Xi}_{33} x_i}{h_p^2(2\overline{\Xi}_{11} x + \bar{e}_{31}\pi)}\right|}\ ;\ i = 1,2,3 \tag{10}$$

And $\mathbf{x_i}$ are the roots of the following cubic equation:

$$\frac{P_0\,\omega^2 h_p^2(2\overline{\Xi}_{11} x + \bar{e}_{31}\pi) - 8\pi\overline{\Xi}_{33}\bar{e}_{31} h_p x^2}{2\pi^2\overline{\Xi}_{33}(d_1 + d_2)x} = \frac{2\pi^2\overline{\Xi}_{33} x}{h_p^2(2\overline{\Xi}_{11} x + \bar{e}_{31}\pi)} \tag{11}$$

In which d_1, d_2 and P_0 are constants depending on geometrical and material properties:

$$P_0 = \rho\,h + 2\,\rho_p h_p\,,\ d_1 = \frac{2\,h^3 E}{3(1-\nu^2)}\,,\ d_2 = \frac{2}{3}\,\overline{C}_{11}^{E}.\left[\left(h + h_p\right)^3 - h^3\right] \tag{12}$$

1. Vibration Analysis of a Variable Thickness Isotropic Kirchhoff Annular Plate Covered with Piezoelectric Layers Using Modified Wave Method.

Developing A State Vector In Terms Of Waves:

A state vector should be constructed in this stage and its elements should be written in form of wave terms. Elements of the state vector consist of the parameters which are used as boundary conditions, and should be linearly independent. So one can form a state vector as bellow:

$$\{z\}_{6*1} = \left\{ w(r,\theta), \frac{\partial w}{\partial r}, \frac{\partial \varphi}{\partial r}, M_{rr}, V_r, \varphi \right\}^t_{6*1} \tag{13}$$

[1] To use wave propagation method, the solution to the isotropic plate coupled with piezo layers should now be re-written in a form adoptable to the wave method, so that the calculation of the propagation, transmission and reflection matrices in the subsequent stages would be possible. The best form for $w(r,\theta,t)$ and $\varphi(r,\theta,t)$ is to convert them in exponential forms. So Bessel functions are to renovate in exponential forms. Using the results of (http://www.efunda.com/math/bessel/bessel.cfm; http://en.wikipedia.org/wiki/Bessel_function; Watson, 1995) and (Olver, F.W.J.; L.C. Maximon, 2010), one can find Bessel functions of first and second kind as follows:

$$J_m(r) = \sqrt{\frac{2}{\pi . r}}\, \cos(r-d)\,, \quad Y_m(r) = \sqrt{\frac{2}{\pi . r}}\, Sin\,(r-d) \qquad d = \frac{(2m+1)\pi}{4} \tag{14}$$

Modified Bessel functions of the first and second kind are related to $J_m(r)$ and $Y_m(r)$ as follows:

$$I_m(r) = i^{-m}.\, J_m(i.r) \qquad K_m(r) = \frac{\pi}{2} i^{m+1}\left(J_m(i.r) + i.Y_m(i.r)\right) \tag{15}$$

Also one can use the following transformation:

$$\frac{1}{\sqrt{\delta_i . r}} = f_i . e^{b_i . r} \tag{16}$$

To be sensible, for a case in which $\lambda_1, \lambda_2 < 0 \;\&\; \lambda_3 > 0$, which happens very often, the computations are shown below. Of course any combination of λ_i has exactly the same computations. In this case, $w(r,\theta)$ and $\varphi(r,\theta)$ become:

$$w(r,\theta) = \{[c_1 J_m(\delta_1 r) + c_4 Y_m(\delta_1 r)] + [c_2 J_m(\delta_2 r) + c_5 Y_m(\delta_2 r)] + [c_3 J_m(\delta_3 r) + c_6 Y_m(\delta_3 r)]\} \cos(m\,\theta) \tag{17}$$

$$\varphi(r,\theta) = \{x_1.[c_1 J_m(\delta_1 r) + c_4 Y_m(\delta_1 r)] + x_2.[c_2 J_m(\delta_2 r) + c_5 Y_m(\delta_2 r)] + x_3.[c_3 J_m(\delta_3 r) + c_6 Y_m(\delta_3 r)]\} \cos(m\,\theta) \tag{18}$$

Substituting equations 14 through 16 into 17 and 18, and at the same time using the exponential form of trigonometric functions, gives $w(r,\theta)$ and $\varphi(r,\theta)$ in form of exponential functions:

$$w(r,\theta) = \left\{\left[c_7 e^{(d_7 . r - d . i)} + c_8 e^{(d_8 . r + d . i)}\right] + \left[c_9 e^{(d_9 . r - d . i)} + c_{10} e^{(d_{10} . r + d . i)}\right] + \left[c_{11} e^{(d_{11} . r - d . i)} + c_{12} e^{(d_{12} . r + d . i)}\right]\right\} \cos(m\,\theta) \tag{19}$$

$$\varphi(r,\theta) = \left\{x_1.\left[c_7 e^{(d_7 . r - d . i)} + c_8 e^{(d_8 . r + d . i)}\right] + x_2.\left[c_9 e^{(d_9 . r - d . i)} + c_{10} e^{(d_{10} . r + d . i)}\right] + x_3.\left[c_{11} e^{(d_{11} . r - d . i)} + c_{12} e^{(d_{12} . r + d . i)}\right]\right\} \cos(m\,\theta) \tag{20}$$

In which c_7 to c_{12} are new constants and d_7 to d_{12} are as follows:

$$d_7 = \delta_1 . i + b_1\,, d_8 = -\delta_1 . i + b_1 \tag{21}$$

$$d_9 = \delta_2 . i + b_2\,, d_{10} = -\delta_2 . i + b_2 \tag{22}$$

$$d_{11} = -\delta_3 + b_3\,, d_{12} = \delta_3 + b_3 \tag{23}$$

With the new form of the solution, one can use wave propagation method and inscribe the solution as a superposition of waves, because the vibration of every structural element may be considered as a waves which propagate forwards and backwards in the structure (Fig.3). The form of the waves depends on the nature of the governing differential equation of the structure.

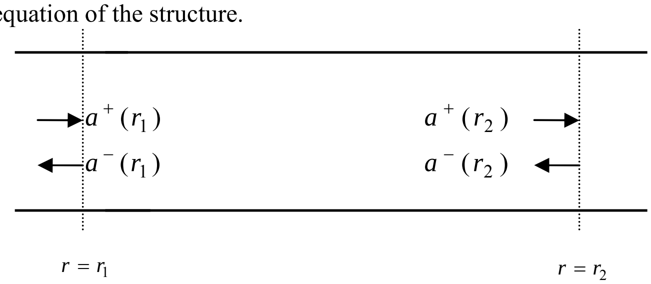

Fig.3 Waves Moving In A Structural Element

So the solution to the Vibration of each segment can be written as:

$$w(r,\theta) = (a_1^+ + a_1^- + a_2^+ + a_2^- + a_3^+ + a_3^-)\cos(m\theta) \tag{24}$$

$$\varphi(r,\theta) = [x_1(a_1^+ + a_1^-) + x_2(a_2^+ + a_2^-) + x_3(a_3^+ + a_3^-)]\cos(m\theta) \tag{25}$$

In which

$$a_1^+ = c_7 e^{(d_7.r-d.i)},\ a_1^- = c_8 e^{(d_8.r+d.i)} \tag{26}$$

$$a_2^+ = c_9 e^{(d_9.r-d.i)},\ a_2^- = c_{10} e^{(d_{10}.r+d.i)} \tag{27}$$

$$a_3^+ = c_{11} e^{(d_{11}.r-d.i)},\ a_3^- = c_{12} e^{(d_{12}.r+d.i)} \tag{28}$$

By now, one can construct elements of the state vector, in terms of the wave terms as follows:

$$\begin{Bmatrix} w \\ \frac{\partial w}{\partial r} \\ \frac{\partial \varphi}{\partial r} \\ M_{rr} \\ V_r \\ \varphi \end{Bmatrix} = \begin{bmatrix} t11 & t12 & t13 & t14 & t15 & t16 \\ t21 & t22 & t23 & t24 & t25 & t26 \\ t31 & t32 & t33 & t34 & t35 & t36 \\ t41 & t42 & t43 & t44 & t54 & t46 \\ t51 & t52 & t53 & t54 & t55 & t56 \\ t61 & t62 & t63 & t64 & t65 & t66 \end{bmatrix} . \begin{Bmatrix} a_1^+ \\ a_1^- \\ a_2^+ \\ a_2^- \\ a_3^+ \\ a_3^- \end{Bmatrix} \tag{29}$$

Or:

$$\{z\}_{6*1} = [t_{ij}].\{a_1^+, a_1^+, a_1^+, a_1^+, a_1^+, a_1^+\}^t \tag{30}$$

The elements of $[t_{ij}]$ should be extracted now. Also for the plate under study here, using 0(Wang *et al.*, 2001) and (Duan *et al.*, 2005), one can take out M_{rr} and V_r as follows:

$$M_{rr} = -[(d_1+d_2)\frac{\partial^2 w}{\partial r^2} + \frac{4}{\pi}h_p\bar{e}_{31}\varphi + (d_1+d_2-2A_1)\left(\frac{\partial^2 w}{r^2\partial\theta^2} + \frac{\partial w}{r\partial r}\right)] \tag{31}$$

$$V_r = q_r + \frac{1}{r}.\frac{\partial M_{r\theta}}{r\partial r} = \left[(d_1+d_2)\frac{\partial}{\partial r}\Delta w - \frac{4}{\pi}h_p\bar{e}_{31}\frac{\partial\varphi}{\partial r}\right] - \frac{2A_1}{r}\left(\frac{\partial^3 w}{r\partial r\partial\theta^2} + \frac{\partial w}{r^2\partial\theta^2}\right) \tag{32}$$

In which d_1, d_2 and A_1 are constants depending on geometrical and material properties:

$$A_1 = \frac{1}{2}\left[(1-v)d_1 + \left(1 - \frac{\overline{C}_{12}^E}{\overline{C}_{11}^E}\right)d_2\right] \tag{33}$$

And also Δ is the Laplacian operator in polar coordinate:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2} \tag{34}$$

Using $w(r,\theta)$, φ and also the relations for M_{rr} and V_r, one can calculate the elements of $[t_{ij}]$ as follows:

$$[t_{ij}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} \\ x_1.d_7 & x_1.d_8 & x_2.d_9 & x_2.d_{10} & x_3.d_{10} & x_3.d_{12} \\ t41 & t42 & t43 & t44 & t45 & t46 \\ t51 & t52 & t53 & t54 & t55 & t56 \\ x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \end{bmatrix} \tag{35}$$

Where:

$$t_{4j} = -\{[(d_1+d_2)\,d_{6+j}^2 + \frac{4}{\pi}h_p\bar{e}_{31}.x_k. \tag{36}$$

$$+(d_1+d_2-2A_1)(\frac{-m^2}{(r+n.L)^2}+\frac{d_{6+j}}{r+n.L})]\}.\cos(m\,\theta);\begin{Bmatrix} for\ j=7,8\ then\ k=1\\ for\ j=9,10\ then\ k=2\\ for\ j=11,12\ then\ k=3\end{Bmatrix}$$

$$t_{5j}=-\{(d_1+d_2)(d_{6+j}^3-\frac{d_{6+j}}{(r+n.L)^2}+\frac{d_{6+j}^2}{r+n.L}+\frac{2\,m^2}{(r+n.L)^3}-\frac{m^2 d_{6+j}}{(r+n.L)^2}+\frac{4}{\pi}h_p\bar{e}_{31}.x_k.d_{6+j})$$

$$+\frac{2A_1 m^2}{(r+n.L)^2}.\left(d_{6+j}-\frac{1}{r+n.L}\right)\}.\cos(m\,\theta);\begin{Bmatrix} for\ j=7,8\ then\ k=1\\ for\ j=9,10\ then\ k=2\\ for\ j=11,12\ then\ k=3\end{Bmatrix} \quad (37)$$

Extracting Propagation, Reflection and Transmission Matrices:

As proved in this paper, for the plate under study, the solution gives 2 pairs of positive and negative wave components as follows:

$$\{a^+\}=\begin{Bmatrix}a_1^+\\a_2^+\\a_3^+\end{Bmatrix}\ and\ \{a^-\}=\begin{Bmatrix}a_1^-\\a_2^-\\a_3^-\end{Bmatrix} \quad (38)$$

Applying the relationship between positive and negative waves at two different points in a segment (Fig.3), the Propagation Matrix can be achieved as follows:

$$\begin{Bmatrix}\{a^+(r_2)\}\\\{a^-(r_1)\}\end{Bmatrix}=\begin{bmatrix}[F^+]_{3*3} & [0]_{3*3}\\ [0]_{3*3} & [F^-]_{3*3}\end{bmatrix}\begin{Bmatrix}\{a^+(r_1)\}\\\{a^-(r_2)\}\end{Bmatrix} \quad (39)$$

where $[F^+]$ and $[F^-]$ are the propagation matrices, which can be obtained as below:

$$[F^+]=\begin{bmatrix}e^{d_7.L} & 0 & 0\\ 0 & e^{d_9.L} & 0\\ 0 & 0 & e^{d_{11}.L}\end{bmatrix} \quad (40)$$

$$[F^-]=\begin{bmatrix}e^{-d_8.L} & 0 & 0\\ 0 & e^{-d_{10}.L} & 0\\ 0 & 0 & e^{-d_{12}.L}\end{bmatrix} \quad (41)$$

Now, to obtain the reflection and Transmission Matrices, one must use the relationship between the state vector in physical domain and the state vector in wave domain using the results of (Mei, 2002; Nikkhah-Bahrami *et al.,* 2008; Loghmani, Masih, 2008) and(Khoshbayani *et al.,* 2011):

$$\{z\}_{6*1}=\begin{bmatrix}[\psi^+]_{3*3} & [\psi^-]_{3*3}\\ [\phi^+]_{3*3} & [\phi^-]_{3*3}\end{bmatrix}\begin{Bmatrix}\{a^+\}_{3*1}\\\{a^-\}_{3*1}\end{Bmatrix} \quad (42)$$

$[\psi^\pm]_{3*3}$ and $[\phi^\pm]_{3*3}$ are a so-called displacement and internal force matrices, and their elements can be extracted using the elements of $[t_{ij}]$ as follows:

$$[\psi^+]_{3*3}=\begin{bmatrix}t11 & t13 & t15\\ t21 & t23 & t25\\ t31 & t33 & t35\end{bmatrix} \quad (43)$$

$$[\psi^-]_{3*3}=\begin{bmatrix}t12 & t14 & t16\\ t22 & t24 & t26\\ t32 & t34 & t36\end{bmatrix} \quad (44)$$

$$[\phi^+]_{3*3}=\begin{bmatrix}t41 & t43 & t45\\ t51 & t53 & t55\\ t61 & t63 & t65\end{bmatrix} \quad (45)$$

$$[\phi^-]_{3*3}=\begin{bmatrix}t42 & t44 & t46\\ t52 & t54 & t56\\ t62 & t64 & t66\end{bmatrix} \quad (46)$$

where all the elements of $[\psi^\pm]_{3*3}$ and $[\phi^\pm]_{3*3}$, i.e. elements of t_{ij}, are defined in the previous section.

When waves encounter boundary conditions or discontinuities at the end of each segment, some are transmitted and some are reflected (Fig.4).

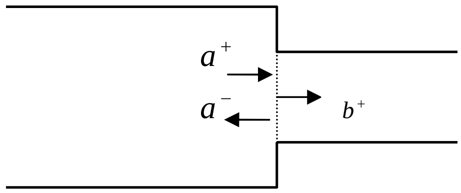

Fig.4 Reflection And Transmission Of Waves In Step Rod

Fig.5 Relations Between Propagated, Reflected And Transmitted Waves Are Defined As Follow:

$$\{a^-\} = [R].\{a^+\} \quad (47)$$

$$\{b^+\} = [T].\{a^+\} \quad (48)$$

in which $[R]$ and $[T]$ are reflection and transmission matrices. Using the last two equations relations and considering the condition of equilibrium and continuity at the step, the transmission and reflection matrices would be extracted as follows 0(Khoshbayani *et al.*, 2011):

$$[R] = -[-\phi_b^+.(\psi_b^+)^{-1}.\psi_a^- + \phi_a^-]^{-1}.\{-\phi_b^+.(\psi_b^+)^{-1}.\psi_a^+ + \phi_a^+\} \quad (49)$$

$$[T] = [\phi_a^-.(\psi_a^-)^{-1}.\psi_b^+ - \phi_b^{+\,-1}].[\phi_a^-.(\psi_a^-)^{-1}.\psi_a^+ - \phi_a^+] \quad (50)$$

In which subscripts 'a' and 'b' indicate the properties of the cross sections before and after each step. Matrices $[R]$ and $[T]$ are forward reflection and transmission matrices and if one exchange the subscripts 'a' and 'b', the backward reflection and transmission matrices would be achieved.

Modified Wave Method to the Variable Thickness Assembly

Using (Nikkhah-Bahrami *et al.*, 2008; Loghmani, Masih., 2008) and 0(Khoshbayani *et al.*, 2011), one can write the waves in subsequent step in terms of the waves at the first step, so the solution of each segment can be written in terms of the waves at the first edge. To clarify the method, for a one step part, i.e. Fig.6 , one can write 0(Khoshbayani *et al.*, 2011):

$$\begin{Bmatrix} a_1^+ \\ a_1^- \end{Bmatrix} \begin{bmatrix} \mu_1 & \lambda_1 \\ \eta_1 & \beta_1 \end{bmatrix} \begin{Bmatrix} a_0^+ \\ a_0^- \end{Bmatrix} \quad (51)$$

Where:

$$\mu_{1=} [T_{f1} - R_{b1}.T_{b1}^{-1}.T_{f1}].F^+(L_1) \quad (52)$$

$$\lambda_1 = (R_{b1} - T_{b1}^{-1}).(F^-(L_1))^{-1} \quad (53)$$

$$\eta_1 = -T_{b1}^{-1}.R_{f1}.F^+(L_1) \quad (54)$$

$$\beta_1 = T_{b1}^{-1}.(F^-(L_1))^{-1} \quad (55)$$

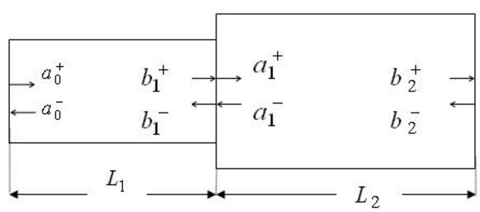

Fig.6 Reflection And Transmission Of Waves At A Step

Using the above mentioned methodology, for all steps and segment of a non-uniform waveguide with an arbitrary variable in the cross-section, the positive and negative traveling waves at the right boundary condition are obtained in te rms of waves at the left boundary condition as follows:

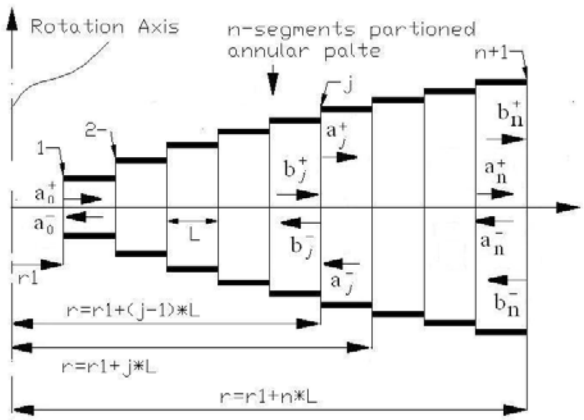

Fig.7 Traveling waves in an arbitrary variable thickness plate

$$\begin{bmatrix} a_n^+ \\ a_n^- \end{bmatrix} = \begin{bmatrix} \mu_n & \lambda_n \\ \eta_n & \beta_n \end{bmatrix} \begin{bmatrix} \mu_{n-1} & \lambda_{n-1} \\ \eta_{n-1} & \beta_{n-1} \end{bmatrix} \begin{bmatrix} \mu_{n-2} & \lambda_{n-2} \\ \eta_{n-2} & \beta_{n-2} \end{bmatrix} \cdots \begin{bmatrix} \mu_1 & \lambda_1 \\ \eta_1 & \beta_1 \end{bmatrix} \begin{bmatrix} a_0^+ \\ a_0^- \end{bmatrix} \tag{56}$$

Or:

$$\begin{bmatrix} a_n^+ \\ a_n^- \end{bmatrix} = \begin{bmatrix} \mu_{total} & \lambda_{total} \\ \eta_{total} & \beta_{total} \end{bmatrix} \begin{bmatrix} a_0^+ \\ a_0^- \end{bmatrix} \tag{57}$$

Where for any arbitrary segment one can write:

$$\mu_i = \left[T_{fi} - R_{bi}.T_{bi}^{-1}.R_{fi} \right].F^+(L_i) \tag{58}$$

$$\lambda_i = (R_{bi} - T_{bi}^{-1}).(F^-(L_i))^{-1} \tag{59}$$

$$\eta_i = -T_{bi}^{-1}.R_{fi}.F^+(L_i) \tag{60}$$

$$\beta_i = T_{bi}^{-1}.(F^-(L_i))^{-1} \tag{61}$$

At last a propagation matrix, relates the outer edge to the last step, i.e. $\{b_n^{\pm}\}$ to $\{a_n^{\pm}\}$. Relations between propagated waves and entrance waves in the "nth" segment are:

$$b_n^+ = F^+(L_n).a_n^+ \tag{62}$$

$$b_n^- = (F^-(L_n))^{-1}.a_n^- \tag{63}$$

in which, L_n is the length of "nth" segment.

Now it is possible to obtain positive and negative waves in the right boundary condition in terms of waves in the left boundary condition.

Applying Boundary Conditions:

Satisfying the boundary conditions, one would yield a system of homogeneous linear equations. The nontrivial roots of the characteristic equation of these simultaneous linear equations are the natural frequencies of the plate. Different types of boundary conditions are as follows:

a) At the free edge:

$$M_{rr} = V_r = \frac{\partial \varphi}{\partial r} = 0 \tag{64}$$

b) At the simply supported edge:

$$w = M_{rr} = \frac{\partial \varphi}{\partial r} = 0 \tag{65}$$

c) At the clamped edge:

$$w = \frac{\partial w}{\partial r} = \frac{\partial \varphi}{\partial r} = 0 \tag{66}$$

Case Study:

In order to verify the method presented in this paper, case studies are performed. At first, for a constant thickness plate, the results of modified wave method developed in this paper, are compared to the results of 0(Duan *et al.,* 2005), in which analytical and F.E. methods are used to analyze the problem(0). After this verification, which shows very close and accurate results for MWM, a nonlinear variable thickness plate is considered and analyzed (00), for which its results are compared to that of achieved via F.E. Again for the case of variable thickness plate, the results of MWM, are very close to F.E. results. In this study, the geometrical radial variation of the variable thickness plate is defined as this general function:

$$h(r) = hi.\left(1 + \left(\frac{r - r_i}{r_o - r_i}\right)^p \left(\frac{h_o}{h_i} - 1\right)\right) \tag{67}$$

Geometrical and Electro-mechanical properties of the plate are considered as 0 and 0. The results are listed through the coming tables.

Table 1. geometrical properties of the plate 0

$r_o/r_i = 6$	$60mm/10mm$
$h_o/h_i = 1$	$30mm/30mm$
$h_o/h_i = 3$	$30/10$
$h_o/h_i = 0.33$	$10/30$

Table 2. Electro-Mechanical properties of the plate (Duan *et al.*, 2005)

	Isotropic metal	Piezoelectric (PZT4)
E (Gpa)	200	$C^E_{11}=132$ $C^E_{12}=71$ $C^E_{13}=73$ $C^E_{33}=115$
	0.3	0.31
Density (Kg/m^3)	7800	7500
e_{31}(cm^{-2})		-4.1
e_{33}(cm^{-2})		14.1
Ξ_{11}(F/m)		7.124×10^{-9}
$\Xi 33$(F/m)		5.841×10^{-9}

Table 3. Natural frequencies of a constant thickness plate

$h_o/h_i = 1$, $r_o/r_i = 6$, P = arbitrary, $h_p/h_i = 1$

	Mode Type	MWM (Hz)	F.E Method (Hz)	Analytical Method (Hz) 0	Diff (%) between WM and F.E	WM and Analytical	Analytical and F.E. 0
	(0,0)	450.4	447.7	448.24	0.6	0.48	0.1
	(0,1)	1244.4	1219.5	1239.8	2.0	0.37	1.66
B.C.:	(1,0)	471.5	468.47	470.06	0.65	0.31	0.36
Clamp-	(1,1)	1269.7	1239.2	1255	2.4	1.15	1.87
Clamp	(2,0)	560.83	552.7	558.3	1.45	0.45	0.99
	(2,1)	1414.6	1375	1407.6	2.8	0.50	2.38
	(0,0)	294.9	294.3	293.47	0.2	0.49	-0.31
	(0,1)	994.9	981.5	990.5	1.35	0.44	0.92
B.C.: Clamp-	(1,0)	318	315.44	315.76	0.8	0.72	0.10
Simply	(1,1)	1036.2	1016.5	1028.5	1.9	0.74	1.16
	(2,0)	405.5	399.8	403.7	1.4	0.44	0.95
	(2,1)	1161.5	1136	1155.9	2.20	0.48	1.75
	(0,0)	222.9	222.1	222.3	0.8	0.27	0.04
	(0,1)	832.9	823.7	827.7	1.1	0.62	0.49
B.C.:	(1,0)	259	253.66	256.8	2.05	0.85	1.30
Simply-	(1,1)	892.1	874.2	885.1	2	0.78	1.23
Simply	(2,0)	377.3	368.2	375	2.4	0.61	1.84
	(2,1)	1069.4	1038.4	1061.9	2.9	0.70	2.27

Table 4. Natural frequencies of a concave plate (clamp-clamp)

B.C.: Clamp-Clamp , $h_o/h_i = 3$, $r_o/r_i = 6$									
		P = 1			P = 2			P = 4	
Mode Type	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)
(0,0)	458.02	451.2	1.49	421.7	415.5	1.47	386.1	380.5	1.44
(0,1)	1248.2	1222	2.10	1094.4	1072	2.05	977.8	958.3	1.99
(1,0)	491.6	482.2	1.92	447.5	439.04	1.89	404.9	397.4	1.85
(1,1)	1290.8	1262	2.23	1127.8	1103	2.2	1004	982.9	2.1
(2,0)	603.4	591.4	2.0	533	522.45	1.98	469.8	460.67	1.95
(2,1)	1425	1391	2.4	1230.3	1202	2.3	1087.7	1064	2.18

Table 5. Natural frequencies of a convex plate (clamp-clamp)

B.C.: Clamp-Clamp , $h_o/h_i = 0.33$, $r_o/r_i = 6$									
		P = 1			P = 2			P = 4	
Mode Type	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)
(0,0)	429.6	423.4	1.45	467	459.1	1.7	500.1	490.1	2
(0,1)	1201	1177	1.99	1340.8	1310	2.3	1461.8	1426	2.45
(1,0)	442.5	434.05	1.9	437	472.8	2.1	522	509.5	2.4
(1,1)	1229	1205	1.95	1377.3	1347	2.2	1506.7	1469	2.5
(2,0)	504	493.9	2	561.1	549.3	2.1	621.3	604.8	2.65
(2,1)	1336.7	1308	2.15	1513.3	1477	2.4	1666	1621	2.7

Table 6. Natural frequencies of a concave plate (simply-simply)

B.C.: Simply-Simply , $h_o/h_i = 3$, $r_o/r_i = 6$									
		P = 1			P = 2			P = 4	
Mode Type	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)
(0,0)	200	196.5	1.77	164	161.5	1.6	144.2	142	1.5
(0,1)	822.4	805.1	2.1	682.5	669.5	1.9	572.1	562.4	1.7
(1,0)	251.7	246.7	2	210.5	206.7	1.8	180.8	178	1.55
(1,1)	880	860.6	2.2	730.1	715.5	2	613	602	1.8
(2,0)	400.6	391.8	2.2	338.4	331.3	2.1	287	281.4	1.9
(2,1)	1056.3	1032	2.3	875.8	857	2.15	739.6	724.8	2

Table 7. Natural frequencies of a convex plate (simply-simply)

B.C.: Simply-Simply , $h_o/h_i = 0.33$, $r_o/r_i = 6$									
		P = 1			P = 2			P = 4	
Mode Type	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)
(0,0)	238.2	233.4	2	273.1	268.18	1.8	305.3	300.1	1.7
(0,1)	823.4	805.23	2.2	948.3	929.3	2	1062.2	1042	1.9
(1,0)	255	249.6	2.1	293.7	288.1	1.9	334.1	328.2	1.75
(1,1)	868.5	848.5	2.3	1003	982.1	2.1	1123.2	1103	1.8
(2,0)	348.5	340.2	2.4	404.3	395.4	2.2	464.5	456.1	1.8
(2,1)	1027.7	1001	2.6	1190.2	1164	2.2	1334.4	1309	1.9

Table 8. Natural frequencies of a concave plate (free-free)

B.C.: Free-Free , $h_o/h_i = 3$, $r_o/r_i = 6$									
		P = 1			P = 2			P = 4	
Mode Type	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)
(0,0)	114.3	112.4	1.7	96.3	94.5	1.9	82.3	80.5	2.2
(0,1)	555.8	544.7	2	430.1	420.2	2.3	350.7	342	2.5
(1,0)	260	254.5	2.1	214.8	209.5	2.5	183.7	178.8	2.7
(1,1)	726.7	710	2.3	594.3	578	2.75	505.4	491	2.85
(2,0)	489.7	477.7	2.45	402.67	391.4	2.8	336.7	327	2.9
(2,1)	1073.5	1045	2.65	887	860.8	2.95	758.5	735	3.1

Table 9. Natural frequencies of a convex plate (free-free)

B.C.: Free-Free , $h_o/h_i = 0.33$, $r_o/r_i = 6$									
		P = 1			P = 2			P = 4	
Mode Type	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)	Wave Method	F.E	Diff (%)
(0,0)	150	147.3	1.8	173.7	170.9	1.6	186.1	183.3	1.5
(0,1)	587.6	575.9	2	710.3	698.2	1.7	822.5	809.3	1.6
(1,0)	313.3	306.7	2.1	379.4	371.8	2	422.9	415.7	1.7
(1,1)	851.7	833.4	2.15	995.4	975.5	2	1123.8	1104	1.76
(2,0)	494.3	482.9	2.3	599.3	586.7	2.1	691.1	678.7	1.8
(2,1)	1184.6	1155	2.5	1395.1	1363	2.3	1581.6	1550	2

Conclusions:

In order to verify wave method presented in this paper, case studies and related comparisons are done for both constant and variable thickness plates. In all cases, very accurate and consistent results are obtained through modified wave method. For MWM developed here, the sizes of the matrixes remain constant with the increscent of the segments which is a real benefit in comparison with the well-known wave method(Achenbach, J.D., 1973; Doyle, J.F., 1989; Doyle, J.F.,; Kolsky, H., 1963; Harland *et al.,* 2000; Mei C., 2002; Lee *et al.,* 2007; Nikkhah-Bahrami *et al.,*2008; Loghmani, Masih., 2008). Besides this novelty, this method has the advantage that it gives the characteristic equation containing all the natural frequencies as its roots, unlike other approximate methods such as weighted residual, Rayleigh-Ritz and finite difference methods which give limited number of natural frequencies. Also since each segment has exact analytical solution, in contrast to other approximate methods, much higher accuracy is obtained even with fewer numbers of partitions.

In all cases, for concave plates, if the degree of concavity increases, the natural frequencies decrease. Reversely, for convex plates, by increasing the convexity degree, the natural frequencies also increase.

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