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Some New Results About The Generalized Entropic Property

Amir Ehsani

Department of Mathematics, Mahshahr Branch, Islamic Azad University
Mahshahr, Iran.

Abstract. In this paper we define concept of the generalized entropic property for the pair of operations, (f, g), of an algebra, $A=(A,F)$. We investigate the relations between entropic property and the generalized entropic property for the pair of operations of the algebra, *A*.

Key words: Complex algebra, Mode, Entropic algebra, Generalized entropic property.

INTRODUCTION

Given an algebra $A=(A,F)$ we define complex operations for every $\phi \neq A_1,...,A_n \subseteq A$ and every n-ary $f \in F$ on the set $\rho(A)$ of all non-empty subsets of the set A by $f(A_1,...,A_n)=\{ f(a_1,...,a_n): a_i \in A_i \}$. The algebra $Cm\,A=(\rho(A),F)$ is called the complex algebra of *A*.

The complex algebras (called also the globals or the powers of algebras) were studied by several authors (G. Grätzer and H. Lakser (Grätzer G. H Lakser. 1988), C. Brink Brink C., 1993= Burris S., H. P Sankappanavar. 1981, I. Bošnjak and R. Madarász (Bošnjak I and R Madarász. 2003), A. Romanowska and J. D. H. Smith (Romanowska A. J.D.H Smith. 1985- Romanowska A. J.D.H Smith. 2002), K. Adaricheva, A. Pilitowska, D. Stanovský (Adaricheva K. A Pilitowska. D Stanovský. 2006) and others).

The notion of complex operations is widely used. In groups, for instance, a coset *xN* is the complex product of the singleton *{x}* and the subgroup *N*. For a lattice *L,* the set *Id L* of its ideals forms a lattice under the set inclusion. If *L* is distributive, then its joint and meet in *Id L* are precisely the complex operations obtained from the joint and meet of *L,* so *Id L* is a subalgebra of *Cm L.*

Now, consider the set *CSub A* of all (non-empty) subalgebras of algebra *A.* This set may or may not be closed under complex operations. For instance, if A is an abelian group, it is closed; however, for the majority of groups, it is not closed. In the former case, *CSub A* is a subuniverse of *Cm A* and we call it a complex algebra of subalgebras. We will say that *A* has the complex algebra of subalgebras or that *CSub A* exists.

Definition 1.1:

The algebra, $A=(A,F)$, is called entropic (or medial) if it satisfies the identity of mediality:

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm})) = f(g(x_{11},...,x_{1m}),...,g(x_{n1},...,x_{nm})) \quad (1)$$

For every *n-ary* $f \in F$ and *m-ary* $g \in F$.

In other words, the algebra A is medial if it satisfies the hyperidentity of mediality (Movsisyan Yu.M., 1986- Movsisyan Yu.M., 1998). Note that a groupoid is entropic iff it satisfies the identity of mediality (Ježek J. T Kepka. 1983): $xy.uv \approx xu.yv$.

Following (Burris S., H. P Sankappanavar. 1981), the algebra $A=(A,f)$ with one n-ary operation is called a mono-n-ary algebra. It could be entropic iff it satisfies the identity:

$$f(f(x_{11},...,x_{n1}),...,f(x_{1n},...,x_{nn})) \approx f(f(x_{11},...,x_{1n}),...,f(x_{n1},...,x_{nn}))$$

A variety *V* is called entropic (or medial) if every algebra in *V* is entropic.

An algebra, *A,* is called idempotent (commutative), if every operation of *A* is idempotent (commutative).

An n-ary operation *f* is called commutative, if $f(x_1,x_2,...,x_n) = f(x_{\alpha(1)},x_{\alpha(2)},...,x_{\alpha(n)})$, where $\alpha \in S_n$.

The n-ary operation *f* is called idempotent, if $f(x,...,x)=x$.

An idempotent entropic algebra is called a mode (Romanowska A. J.D.H Smith. 2002).

Corresponding Author: Amir Ehsani, Department of Mathematics, Mahshahr Branch, Isamic Azad University Mahshahr, Iran, E-mail: Amirehsany@yahoo.com

Definition 1.2:

Let $A=(A,F)$ be an algebra. Let us define the concept of *m-ary* term operation of the algebra, *A*, by induction:-

1- Every *m-ary* trivial operation, $e_i^m(x_1,...,x_m)=x_i$, on the set, *A*, is an *m-ary* term operation of the algebra, *A*.
2- If $f\in F$ is an *n-ary* operation and $f_1,...,f_n$ are *m-ary* term operations of the algebra, *A*, then $f(f_1,...,f_n)(x_1,...,x_m)=f(f_1(x_1,...,x_m),...,f_n(x_1,...,x_m))$ is an *m-ary* term operation of *A*.
3- There is not another *m-ary* term operation of the algebra, *A*.

2.Tthe Generalized Entropic Property:

Definition 2.1:

We say that a variety *V* (respectively, the algebra *A*) satisfies the generalized entropic property if for every *n-ary* operation *f* and *m-ary* operation *g* of *V* (of *A*) there exist *m-ary* term operations $t_1,...,t_n$ such that in *V* (in *A*) the below lidentity holds (Adaricheva K. A Pilitowska. D Stanovský. 2006).

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm}))=f(t_1(x_{11},...,x_{1m}),...,t_n(x_{n1},...,x_{nm})) \qquad (2)$$

For example, a groupoid satisfies the generalized entropic property, if there are binary terms *t* and *s* such that, $xy.uv\approx t(x,u).s(y,v)$.

It was proved in (Evans T., 1962) that for the variety *V* of groupoids, every groupoid in *V* has the complex algebra of subalgebras iff *V* satisfies the above identity for some *t* and *s*.

Example 2.2:

Let *R* be a ring with a unit, *G* a subgroup of the multiplicative monoid of *R*, and *X* a subset of *G* closed under conjugation by elements of *X* and closed under the maping $x\mapsto 1-x$, where – is the ring subtraction.

If *M* is a left module over the ring *R*, we define for every element $r\in R$ a binary operation $\underline{r}:M^2\to M$ by : $\underline{r}(x,y)=(1-r)x+ry$.

Of course, the groupoid *(M, r)* is idempotent and entropic for every $r\in R$. Now, consider the algebra $\underline{M}$=(M, $\underline{X}$), where $\underline{X}=\{\underline{r}|r\in R\}$. For every $r,t\in X$, we put $s_1=(1-r)^{-1}t(1-r)\in X$ and $s_2=r^{-1}tr\in X$, and we get

$$\underline{t}(\underline{r}(x_1,x_2),\underline{r}(y_1,y_2))\approx(1-t)(1-r)x_1+(1-t)rx_2+t(1-r)y_1+try_2\approx$$
$$(1-r)(1-s_1)x_1+r(1-s_2)x_2+(1-r)s_1y_1+rs_2y_2\approx\underline{r}(\underline{s_1}(x_1,y_1),\underline{s_2}(x_2,y_2)).$$

So, the algebra $\underline{M}$ satisfies the generalized entropic property. On the other hand, it is entropic, iff $rt=tr$ for all $r,t\in X$. To check this put $x_1=y_1=y_2=0$ and $x_2=1$ in the previous identity.

If *R* is a non-commutative division ring, *G* is its multiplicative group and $X=R\setminus\{0,1\}$, then $\underline{M}$ is a non-entropic idempotent algebra satisfying the generalized entropic property.

Theorem 2.3:

Every algebra in a variety *V* has the complex algebra of subalgebras, iff the variety *V* satisfies the generalized entropic property.

Proof:

In (Adaricheva K. A Pilitowska. D Stanovský. 2006).

Theorem 2.4:

An idempotent and commutative mono-n-ary algebra, $A=(A,f)$, satisfying the generalized entropic property is entropic.

Proof:

In (Ehsani A., 2011).

3. The Main Results:

By definitions (2.1) and (1.1), we can say that the algebra, $A=(A,F)$, satisfying the generalized entropic property (or is entropic) iff for every pair operations, (f,g), of the algebra *A,* identity (2) (or (1)) holds.

Now, we investigate that, if some pair of operations, (f,g), of the algebra, *A*, satisfying the generalized entropic property then what results we can obtain for the algebra, $A=(A,F)$?

To achieve this perspective, we define the entropic and the generalized entropic property for the pair of operations.

Definition 3.1:

Let *g* and *f* be *m-ary* and *n-ary* operations on the set, *A.* We say that the pair of the operations, *(f, g),* satisfies the generalized entropic property if there exist *m-ary* term operations, $t_1,...,t_n$, of the algebra, $A=(A,f,g)$, such that the identity (2) holds in the algebra, $A=(A,f,g)$.

The pair of operations *(g, f)* is called entropic (or medial) if identity (1) holds in the algebra, $A=(A,f,g)$. If $f=g$, then we say that the operation, *f*, is entropic.

The results for binary pair of operations and ternary pair of operations obtained in (Ehsani A., 2010-Ehsani A). 2011. In this section, we show our new results for the general case of the pair of operations.

Lemma 3.2:

Let (f,g) be the entropic pair of n-ary operations. If *f* and *g* are idempotent and commutative operations, then $f=g$.

Proof:

Using the idempotency, the entropic and the commutativity properties, we have:

$$\begin{aligned} f(x_1,...,x_n) &= f(g(x_1,...,x_1),g(x_2,...,x_2),...,g(x_n,...,x_n)) \\ &= g(f(x_1,x_2,...,x_n),f(x_1,x_2,...,x_n),...,f(x_1,x_2,...,x_n)) \\ &= g(f(x_1,x_2,...,x_n),f(x_2,...,x_n,x_1),...,f(x_n,x_1,...,x_{n-1})) \\ &= f(g(x_1,x_2,...,x_n),g(x_2,...,x_n,x_1),...,g(x_n,x_1,...,x_{n-1})) \\ &= f(g(x_1,x_2,...,x_n),g(x_1,x_2,...,x_n),...,g(x_1,x_2,...,x_n)) \\ &= g(x_1,x_2,...,x_n), \end{aligned}$$

for every $x_1,x_2,...,x_n \in A$. Thus, $f=g$.

Definition 3.3:

Let $A=(A,F)$ be an algebra and $f,g \in F$. We say that the element, *e*, is the unit for the operation, $f \in F$, if:

$$f(x,e,...,e)=f(e,x,e,...,e)=...=f(e,...,e,x)=x,$$

for every $x \in A$.

The element, *e*, is the unite for the pair of operations, (f,g), if it is the unit for the both operations, *f* and *g*. We say that *e* is a unit for the algebra, $A=(A,F)$, if it is a unit for each operation $f \in F$.

Theorem 3.4:

Let $A=(A,F)$ be an algebra and (f,g) be the pair of n-ary and m-ary operations with the unit element, *e*. If (f,g) satisfies the generalized entropic property, then (f,g) is entropic.

Proof:

The generalized entropic property for the pair of operations, (f,g), says that:

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm})) = f(t_1(x_{11},...,x_{1m}),...,t_n(x_{n1},...,x_{nm})),$$

for some m-ary term operations $t_1,...,t_n$. Therefore, by definition of the unite element and the generalized entropic property we have:

$$\begin{aligned} g(x_1,...,x_m) &= g(f(e,...,e,x_1,e,...,e), f(e,...,e,x_2,e,...,e),..., f(e,...,e,x_m,e,...,e)) \\ &= f(t_1(e,...,e),...,t_i(x_1,...,x_m),...,t_n(e,...,e)) \\ &= f(e,...,e,t_i(x_1,...,x_n),e,...,e) \\ &= t_i(x_1,...,x_n), \end{aligned}$$

for every $1 \le i \le n$.
Hence,

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm})) = f(g(x_{11},...,x_{1m}),...,g(x_{n1},...,x_{nm})).$$

Lemma 3.5:
Let (f,g) be an entropic pair of n-ary operations with the unit element, *e*, then $f = g$.

Proof:
By definition of the unit element, *e*, for the pair of operations and the entropic property for the pair of operations, we have:

$$\begin{aligned} g(x_1,...,x_n) &= g(f(x_1,e,...,e), f(e,x_2,e,...,e),..., f(e,...,e,x_n)) \\ &= f(g(x_1,e,...,e),g(e,x_2,e,...,e),...,g(e,...,e,x_n)) \\ &= f(x_1,...,x_n), \end{aligned}$$

for every $x_1,...,x_n \in A$. Thus, $f = g$.

Corollary 3.6:
Every n-ary algebra, $A = (A,F)$, with the unit element, satisfying the generalized entropic property, is the mono-n-ary entropic algebra.

Proof:
Let *e* be the unit element of the algebra *A*, then *e* is the unit element of every pair of operations, (f,g), of the algebra, *A*. So, by the theorem (3.3), every pair of n-ary operations, (f,g), of the algebra, *A*, satisfying the generalized entropic property is entropic. But in this case, by the previous lemma, we have: $f = g$. Thus *A* is a mono-n-ary entropic algebra.

The generalized entropic property for the algebra, $A = (A,f,g)$, with one n-ary and one m-ary operations (respectively *f, g*) means that, the following identities hold:

$$f(f(x_{11},...,x_{n1}),...,f(x_{1n},...,x_{nn})) = f(t_1(x_{11},...,x_{1n}),...,t_n(x_{n1},...,x_{nn})),$$

$$f(g(x_{11},...,x_{m1}),...,g(x_{1n},...,x_{mn})) = g(s_1(x_{11},...,x_{1n}),...,s_m(x_{m1},...,x_{mn})),$$

$$g(g(x_{11},...,x_{m1}),...,g(x_{1m},...,x_{mm})) = g(r_1(x_{11},...,x_{1m}),...,r_m(x_{m1},...,x_{mm})),$$

$$g(f(x_{11},...,x_{n1}),...,f(x_{1n},...,x_{nm})) = f(u_1(x_{11},...,x_{1m}),...,u_n(x_{n1},...,x_{nm})).$$

Immediate consequences of the generalized entropic property in the idempotent algebra, $A = (A,f,g)$, with one n-ary and one m-ary operation (respectively *f*, g) are the following identities, that can be treated as laws of pseudo-distributivity:

$$g(s_1(x_{11},...,x_{1n}),x_{21},...,x_{m1})) = f(g(x_{11},x_{21},..,x_{m1}),g(x_{12},x_{21},..,x_{m1})...,g(x_{1n},x_{21},..,x_{m1}))$$

$$g(x_{11},s_2(x_{21},...,x_{2n}),x_{31},...,x_{m1}) = f(g(x_{11},x_{21},..,x_{m1}),g(x_{11},x_{22},..,x_{m1})...,g(x_{11},x_{2n},..,x_{m1}));$$

$$g(x_{11},...,x_{(m-1)1},s_m(x_{m1},...,x_{mn})) = f(g(x_{11},...,x_{(m-1)1},x_{m1}),...,g(x_{11},...,x_{(m-1)1},x_{mn}))$$

And the entropic low for the algebra, $A = (A,f,g)$, with one n-ary and one m-ary operation (respectively *f*, g) means the following identities:

$$f(f(x_{11},\ldots,x_{n1}),\ldots,f(x_{1n},\ldots,x_{nn})) = f(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{n1},\ldots,x_{nn})), \qquad (3)$$
$$g(g(x_{11},\ldots,x_{m1}),\ldots,g(x_{1m},\ldots,x_{mm})) = g(g(x_{11},\ldots,x_{1m}),\ldots,g(x_{m1},\ldots,x_{mm})), \qquad (4)$$
$$f(g(x_{11},\ldots,x_{m1}),\ldots,g(x_{1n},\ldots,x_{mn})) = g(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{m1},\ldots,x_{mn})). \qquad (5)$$

Theorem 3.7:

Let $A=(A,f,g)$ be an idempotent algebra with one n-ary and one m-ary operation (respectively *f,* g). If *g* is commutative and the pair of operations, (f,g), satisfies the generalized entropic property, then (f,g) is entropic.

Proof:

To prove identity (5), we have from the generalized entropic property:

$$f(g(x_{11},\ldots,x_{m1}),\ldots,g(x_{1n},\ldots,x_{mn}))=g(s_1(x_{11},\ldots,x_{1n}),\ldots,s_m(x_{m1},\ldots,x_{mn})).$$

Using the pseudo-distributiveness and the commutativity of *g*, we obtain:

$$g(s_1(x_{11},\ldots,x_{1n}),x_{21},\ldots,x_{m1}))=f(g(x_{11},x_{21},..,x_{m1}),g(x_{12},x_{21},..,x_{m1}),\ldots,g(x_{1n},x_{21},..,x_{m1}))$$
$$=f(g(x_{21},..,x_{m1},x_{11}),g(x_{21},..,x_{m1},x_{12}),\ldots,g(x_{21},..,x_{m1},x_{1n}))$$ Simil
$$=g(x_{21},..,x_{m1},s_m(x_{11},\ldots,x_{1n})).$$

arly, we can show that:

$$g(s_1(x_{11},\ldots,x_{1n}),x_{21},\ldots,x_{m1}))=g(x_{m1},s_2(x_{11},\ldots,x_{1n}),x_{21},\ldots,x_{(m-1)1}))$$
$$=\cdots=g(x_{21},..,x_{m1},s_m(x_{11},\ldots,x_{1n})).$$

By idempotency and the above identities, we have:

$$s_1(x_{11},\ldots,x_{1n})=g(s_1(x_{11},\ldots,x_{1n}),\ldots,s_1(x_{11},\ldots,x_{1n}))$$
$$=g(s_1(x_{11},\ldots,x_{1n}),s_2(x_{11},\ldots,x_{1n}),\ldots,s_1(x_{11},\ldots,x_{1n}))$$
$$=\cdots=g(s_1(x_{11},\ldots,x_{1n}),s_2(x_{11},\ldots,x_{1n}),\ldots,s_m(x_{11},\ldots,x_{1n}))$$
$$=f(x_{11},\ldots,x_{1n}).$$

In the same manner, we have:

$$s_2(x_{21},\ldots,x_{2n})=f(x_{21},\ldots,x_{2n}),$$
$$\vdots$$
$$s_m(x_{m1},\ldots,x_{mn})=f(x_{m1},\ldots,x_{mn}).$$

Thus, from the generalized entropic property and the above identities, we have:

$$f(g(x_{11},\ldots,x_{m1}),\ldots,g(x_{1n},\ldots,x_{mn}))=g(s_1(x_{11},\ldots,x_{1n}),\ldots,s_m(x_{m1},\ldots,x_{mn}))$$
$$=g(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{m1},\ldots,x_{mn})).$$

Corollary 3.8:

Every idempotent and commutative algebra, $A=(A,f,g)$, with one n-ary and one m-ary operation, satisfying the generalized entropic property, is entropic.

Proof:

Let us show that the identities (3), (4) and (5) hold in the algebra, *A*.
Identities (3) and (4) are proved through the theorem 2.4, and the identity (5) is proved through the theorem 3.7.

Corollary 3.9:

Every idempotent and commutative n-ary algebra, $A=(A,F)$, satisfying the generalized entropic property, is mono-n-ary entropic algebra.

Proof:

It is sufficient to consider theorem 2.4 and lemma 3.2.

4. Summary:

We define the Entropic and the generalized entropic property for the pair of operations, (f,g), of the algebra, $A=(A,F)$, and the following result obtained:

1) For every pair of n-ary and m-ary operations, (f,g), with the unit element, *e*. If (f,g) satisfies the generalized entropic property, then (f,g) is entropic.
2) Every n-ary algebra, $A=(A,F)$, with the unit element, satisfying the generalized entropic property, is the mono-n-ary entropic algebra.
3) In every idempotent algebra, $A=(A,f,g)$, with one n-ary and one m-ary operation, if g is commutative and the pair of operations, (f,g), satisfies the generalized entropic property, then (f,g) is entropic.
4) Every idempotent and commutative algebra, $A=(A,f,g)$, with one n-ary and one m-ary operation, satisfying the generalized entropic property, is entropic.
5) Every idempotent and commutative n-ary algebra, $A=(A,F)$, satisfying the generalized entropic property, is mono-n-ary entropic algebra.

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