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On Extremal Value Problems and Friendly Index Sets of Maximal Outerplanar Graphs

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Abstract: Let G be a graph with vertex set V(G) and edge set E(G) and let A be an abelian group. A labeling f: V(G) → A induces an edge labeling f^* : E(G) → A defined by $f^*(xy) = f(x) + f(y)$, for each edge xy ∈ E(G). For i ∈ A, let $v_f(i)$ = card{v ∈ V(G) : f(v) = i} and $e_f(i)$ = card{e ∈ E(G) : $f^*(e)$ = i}. Let c(f)= {| $e_f(i) - e_f(j)$| : (i,j) ∈ AxA}. A labeling f of a graph G is said to be A-*friendly* if | $v_f(i) - v_f(j)$ | ≤ 1 for all (i,j) ∈ AxA . If c(f) is a (0,1)-matrix for an A-friendly labeling f, then f is said to be A-cordial. When A = Z_2, the ***friendly index set*** of the graph G, FI(G), is defined as {|$e_f(0) - e_f(1)$| : the vertex labeling f is Z_2-friendly}. Let G be a maximal outerplanar graph of order n and f a Z_2-friendly labeling. In this paper, we (a) solve some extremal value problems, and (b) completely determined the friendly index sets for maximal outerplanar graphs with certain topological structures. Particularly, we showed that the friendly indices of maximal outerplanar graphs may not form an arithmetic progression.

Key words: vertex labeling, friendly labeling, cordiality, friendly index set, maximal outerplanar.

INTRODUCTION

Let G be a graph with vertex set V(G) and edge set E(G). Let A be an abelian group. A labeling f: V(G) → A induces an edge labeling f^* : E(G) → A defined by $f^*(xy) = f(x) + f(y)$, for each edge xy ∈ E(G). For i ∈ A, let $v_f(i)$ = card{v ∈ V(G) : f(v) = i} and $e_f(i)$ = card{e ∈ E(G) : $f^*(e)$ = i}. Let c(f)= {| $e_f(i) - e_f(j)$ |: (i,j) ∈AxA}. A labeling f of a graph G is said to be A-*friendly* if |$v_f(i) - v_f(j)$ | ≤ 1 for all (i,j) ∈ A×A. If c(f) is a (0,1)-matrix for an A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey (1991), who generalized the concept of cordial graphs of Cahit (1987). Cahit considered A = $\mathbf{Z}_2$ and he proved the following: every tree is cordial; K_n is cordial if and only if n ≤ 3; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if n ≠ 3 (mod 4); C_n is cordial if and only if n ≠ 2 (mod 4); and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Benson and Lee (1989) showed a large class of cordial regular windmill graphs which include the friendship graphs as a subclass.

Lee and Liu (1991) investigated cordial complete k-partite graphs. Kuo, Chang and Kwong (1997) determined all m and n for which mK_n is cordial. Ho, Lee and Shee (1990) investigated the construction of cordial graphs by Cartesian product and composition. Seoud and Abdel (1999) proved certain cylinder graphs are cordial. Several constructions of cordial graphs were proposed in (Cahit, 1990; Kirchherr, 1991; Kirchherr, 1993; Seah, 1991; Shee and Ho, 1993; 1996). For more details of known results and open problems on cordial graphs, see (Cahit, 1990; Frucht, 1970).

In this paper, we will exclusively focus on A = Z_2, and drop the reference to the group. In [6] the following concept was introduced.

Definition 1:

A friendly index set FI(G) of a graph G is defined as {|$e_f(0) - e_f(1)$| : the vertex labeling f is friendly}.

When the context is clear, we will drop the subscript f. Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality.

Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus in general it is difficult to determine the friendly index sets of graphs. For more known results and open problems on friendly index sets of graphs, see (Y.S. Ho, 2007; Kwong, 2008; Sin-Min Lee, 2008; Lee and Ng, 2007, 2010; W.C. Shiu, 2008).

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In (Lee and Ng, 2008), the friendly index sets of a few classes of graphs, in particular, complete bipartite graphs and cycles are determined. The following result was established.

Theorem 1.1:

For any graph with q edges, the friendly index set $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even and $FI(G) \subseteq \{1, 3, 5, \ldots, q\}$ if q is odd.

Example 1:

Figure 1 illustrates the friendship index set of windmill graph WM_5. The friendship index set of $T(WM_5)$ is $\{1,3,5\}$.

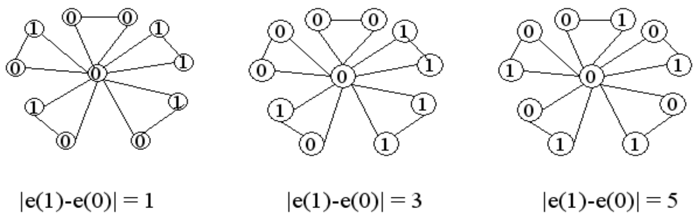

Fig. 1: Some friendly labeling of $T(WM_5)$.

Cubic graphs are 3-regular graphs. In 1989, the Ho, Lee and Shee completely characterized cordial generalized Petersen graphs. The smallest cubic graph is K_4 which has $FI(K_4) = \{2\}$. There are two non-isomorphic cubic graphs of order 6, i.e. $K_{3,3}$ and prism $C_3 x K_2$. Their friendship index sets are distinct.

Example 2:

$FI(K_{3,3}) = \{1,9\}$ and $FI(C_3 x K_2) = \{1,3,5\}$

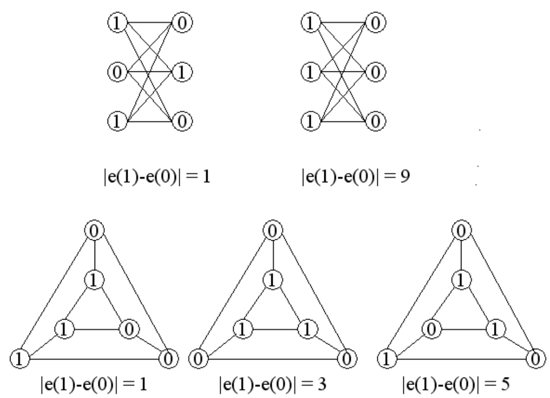

Fig. 2: Some friendly labeling of $K_{3,3}$ and C_3 x K_2.

A planar graph is a graph which can be drawn without edge-crossing. A planar graph *G* is **outer-planar** if there is an embedding of *G* on the plane in which every vertex lies on the exterior face. If we consider a planar graph with no loops or faces bounded by two edges (digons), it may be possible to add a new edge to the presentation of G such that these properties are preserved. When no such adjunction can be made, the graph is called a ***maximal outerplanar graph*** (or MOP) since any additional edge will destroy its outer planar property. We record some facts about maximal outerplanar graphs. The reader is referred to (Kumar and Madhavan, 1989) for details of the proof of the following lemma.

Lemma 1.1.:

Let G be a maximal outerplanar graphs with n vertices, $n \geq 3$. Then:

(i) There are 2n-3 edges, of which there are n-3 chords;

(ii) There are n-2 inner faces, each of which is a cycle C_3;

(iii) There are at least two vertices with degree 2.

Let G be a maximal outer planar graph of order n. Denote by $e^O(i)$ (respectively, $e^I(i)$) the number of outer (respectively, inner) edges of G with label i for i = 0, 1. Clearly, $|e(0) – e(1)| = |e^O(0) – e^O(1) + e^I(0) – e^I(1)|$. In this paper, we (a) solve some extremal value problems, and (b) completely determined the friendly index sets for maximal outerplanar graphs with certain topological structures.

In (Lee & Ng, 2008), the following results were obtained.

Lemma 1.2.:
Any labeling of a cycle will have e(1) equal an even number.

Lemma 1.3.:
A friendly labeling of a cycle must have $e(1) \geq 2$.

Corollary 1.1.:
A friendly labeling of a MOP must have $e^O(1) \geq 2$ and is even.

Theorem 1.2.:
The Friendly Index set of a cycle is given as follows:
(i) $FI(C_{2n}) = \{0, 4, 8, ..., 2n\}$ if n is even.
$FI(C_{2n}) = \{2, 6, 10, ..., 2n\}$ if n is odd.
(ii) $FI(C_{2n+1}) = \{1, 3, 5, ..., 2n – 1\}$.

In what follows, we let $x \in \{0, 1\}$ and the outer edges of a maximal outerplanar graph of order n be (c_1, c_2), (c_2, c_3), ..., (c_{n-1}, c_n), (c_n, c_1), unless otherwise specified . Call the labeling where the vertices c_1 to c_n are labeled alternately with x and 1 – x (x = 0, 1) the *standard labeling*.

2. Extremal Values of FI(MOP):

Theorem 2.1.:
Suppose G is an MOP of order n with the standard labeling.
(i) min {e(0)} = 1 if and only if n = 4;
(ii) min {e(0)} = 2 if and only if n = 5 or n = 6 with G having exactly two degree 2 vertices;
(iii) min $\{e(0)\} \geq 3$ otherwise.

Proof:
If n = 4 or 5, the sufficiency of (i) and (ii) (when n = 5) obviously hold. Suppose n = 6. By Lemma 1.1(iii), G has at least two degree 2 vertices whose end-vertices induced two inner 0-edges. Hence, $e(0) \geq 2$. It is routine to check the 3 possible graphs of G that the sufficiency of (ii) (when n = 6) holds. To complete the proof, we just need to show that (iii) holds.

Let $n \geq 8$ be even. If G has at least 3 degree 2 vertices, then we have $e(0) = e^I(0) \geq 3$. Supppose G has exactly two degree 2 vertices, then G must be a graph with c_1 adjacent to all other vertices. Hence, edge (c_1,c_j) is a 0-edge for all odd j. Therefore, $e^I(0) \geq 3$.

Suppose $n \geq 7$ is odd, then $f(c_1) = f(c_n) = x$ with $e^O(0) = 1$. Since G has at least two degree 2 vertices, at least one of them has two neighbors that induce an inner 0-edge. So, $e(0) \geq 2$. If e(0) = 2, then $e^I(0) = 1$. Note that G has at least 8 distinct pairs of 0- or 1-vertices. If only a pair of 0- or 1-vertices are adjcent in G, then G has an induced C_k, $k \geq 4$. Thus, $e(0) \geq 3$. The proof is thus complete. □

Theorem 2.2:
Let G be a MOP of order n, then max $\{FI(G)\} \leq 2n – 5$. Moreover,
(i) max {FI(G)} = 2n – 5 if and only if n = 4,
(ii) max {FI(G)} = 2n – 7 if and only if n = 5 or 6 with G has exactly two degree 2 vertices, and
(iii) there are infinitely many graphs G with $n \geq 7$ and max {FI(G)} = 2n – 9.

Proof:
To determine max {FI(G)}, we seek to minimize e(0) or e(1). By Corollary 1.1, $e^O(1) \geq 2$ and is even. If $e(0) = 0 = e^O(0)$, then we have standard labeling with n even. By Theorem 1.1, $\min\{e(0)\} \geq 1$, a contradiction. Therefore, $\min\{e(0), e(1)\} \geq 1$. Therefore, max $\{FI(G)\} \leq 2n – 5$. By Theorem 1.1, the equality holds if and only if n = 4. This is attained if $f(c_1) = f(c_3) = x$ and $f(c_2) = f(c_4) = 1 – x$. Thus, (i) holds.

Suppose n = 5. Recall that $e^O(1) \geq 2$ and is even. Thus, $e^O(0) \geq 1$ and is odd. If $e^O(1) = 2$, we have $e(0) \geq 3$ and the five vertices must be labeled with 0 , 0, 1, 1, 1 (or their complements). It is easy to check that $e(1) \geq 3$. If

$e^O(0) = 1$, we have standard labeling. By Theorem 1.1, $\min\{e(0)\} = 2$. Thus, $\min\{e(0), e(1)\} = 2$. Suppose $n = 6$. We now have $e^O(1) \geq 2$ and $e^O(0) \geq 0$, with both are even. If $e^O(0) = 0$, we have standard labeling. By Theorem 1.1, $e(0) \geq 2$. Thus, $\min\{e(0), e(1)\} = 2$. Therefore, the sufficiency of (ii) holds.

To prove the necessity of (ii), we only need to show that $\min\{e(0), e(1)\} \geq 3$ if $n \geq 7$. Suppose $n \geq 7$ is odd. Since $e^O(1) \geq 2$ is even, we have $e^O(0) \geq 1$ is odd. If $e^O(0) = 1$, we have standard labeling with $e(1) \geq 6$. By Theorem 1.1, $e(0) \geq 3$. If $e^O(0) = 3$, then $e^O(1) \geq 4$, as required. If $e^O(0) \geq 5$, then we only need to consider $e^O(1) = 2$. This gives us the labeling sequence 0, 0, ..., 0, 1, 1, 1, ..., 1 (or their complements with $|v(0) - v(1)| = 1$). If $e^I(1) = 0$, then G has an induced C_4, a contradiction. Thus, $\min \{e(1)\} \geq 3$.

Suppose $n \geq 8$ is even. If $e^O(0) = 0$, we have standard labeling with $e(1) \geq 8$. By Theorem 1.1, $e(0) = e^I(0) \geq 3$, as required. If $e^O(0) = 2$, the vertices must be labeled with 1, 0, 1, 0, ..., 1, 0, 0, 1, 0, 1, ..., 0, 1 (or their complements) with $e(1) \geq 6$. If $e^I(0) = 0$, then G has an induced C_k, $k \geq 4$, a contradiction. Hence, $e(0) \geq 3$. We now consider $e^O(0) \geq 4$. Since $e^O(1) \geq 2$ is even, we only need to consider $e^O(1) = 2$. This gives us the labeling sequence 0, 0, ..., 0, 1, 1, 1, ..., 1 (or their complements with $|v(0) - v(1)| = 0$). If $e^I(1) = 0$, then G has an induced C_4, a contradiction. Thus, $\min \{e(1)\} \geq 3$.

To complete the proof, we let $n = \lfloor p/2 \rfloor$. Suppose (c_1, c_p), (c_1, c_{p+1}) and (c_{p+1}, c_n) are 3 inner edges of G. Let $f(c_i) = x$ for $i = 1, 2, 3, ..., p$, and $f(c_i) = 1 - x$ otherwise. Then f is a friendly labeling with $v(x) = p$ and $v(1 - x) = p$ or $p + 1$. Clearly, $e(0) > e(1) = 3$. Thus, $\max \{FI(G)\} = 2n - 9$ is attained and (iii) holds. □

For $p \geq q \geq 3$, let G(p, q) be the class of MOPs of order $p + q$ such that $V(G(p, q)) = \{c_1, c_2, ..., c_p, c_{p+1}, ..., c_{p+q}\}$ and $E(G(p, q)) = \{(c_1, c_3), (c_1, c_4), ..., (c_1, c_p), (c_1, c_{p+1}), (c_{p+1}, c_{p+3}), (c_{p+1}, c_{p+4}), ..., (c_{p+1}, c_{p+q})\} \cup E(C_{p+q})$. From the proof of Theorem 2.2(iii), $\max \{FI(G(p,q))\} = 2(p + q) - 9$ if $p + q \geq 7$.

Theorem 2.3:

Let $G = G(p,p)$ with $p \geq 6$ and is even, then $\max\{FI(G)\} = 2n - 9$ and $2n - 11 \notin FI(G(p,p))$.

Proof:

It follows from Theorem 2.2 that $\max\{FI(G)\} = 2n - 9$. Suppose $2n - 11 \in FI(G)$, then G admits a friendly labeling with $e(1) = 4$ or $e(0) = 4$. Suppose $e(1) = 4$. By Corollary 1.1, $e^O(1) = 2$ or 4.

If $e^O(1) = 2$, then $e^I(1) = 2$. It follows that C_{2p} has p consecutive vertices labeled with x and the remaining vertices labeled with $1 - x$. Recall that the inner edges are (c_1,c_3), (c_1,c_4), ..., (c_1,c_p), (c_1,c_{p+1}), (c_{p+1},c_{p+3}), (c_{p+1},c_{p+4}), ..., (c_{p+1},c_{2p}). If $f(c_1) = f(c_2) = ... = f(c_p) = x$, we only get (c_1,c_{p+1}) as inner 1-edge. So, $e(1) = 3$, a contradiction. If $f(c_2) = f(c_3) = ... = f(c_{p+1}) = x$, we have (c_1,c_i), $i = 3, 4, ..., p+1$, are inner 1-edges. Since $p \geq 6$, $e^I(1) \geq 5$, a contradiction. Otherwise, $f(c_i) = f(c_{i+1}) = ... = f(c_{i+p-1}) = x$ and $f(c_{i+p}) = f(c_{i+p+1}) = ... = f(c_{2p}) = f(c_1) = f(c_2) = ... = f(c_{i-1}) = 1 - x$ for $i \in \{3, 4, ..., p\}$. Hence, we have (c_1, c_k) and (c_{p+1},c_j) are inner 1-edges for $k = i, i + 1, ..., p+1$ and $j = i + p, i + p + 1, ..., 2p$. Since $p \geq 6$, $e^I(1) \geq 3$, also a contradiction.

If $e^O(1) = 4$, then $e^I(1) = 0$. Suppose $f(c_1) = x$, then $f(c_i) = x$ for $i = 3, 4, ..., p, p+1$. This implies that $f(c_j) = x$ for $j = p+3, p+4, ..., 2p$. However, this labeling is not friendly, a contradiction.

Suppose $e(0) = 4$. If $e^O(0) = 0$, then $e^O(1) = 2p$ and we get standard labeling with $e^I(0) = 4$. Obviously, (c_1,c_i) and (c_{p+1},c_j) are inner 0-edges for $i = 3, 5, 7, ..., p + 1$ and $j = p+3, p+5, ..., 2p - 1$. Since $p \geq 6$, $e^I(0) \geq 5$, a contradiction.

If $e^O(0) = 2$, then $e^I(0) = 2$. By symmetry, we consider the following 3 cases.

Case (a): (c_1,c_i) and (c_1,c_j) are two inner 0-edges for $3 \leq i < j \leq p$. In this case, $f(c_1) = f(c_i) = f(c_j) = 1 - f(c_{p+1}) = f(c_k)$ for $k = p+3, p+4, ..., 2p$. This implies that $e^O(0) > 2$, a contradiction.

Case (b): (c_1,c_i) and (c_1,c_{p+1}) are two inner 0-edges for $3 \leq i \leq p$. In this case, $f(c_1) = f(c_i) = f(c_{p+1}) = 1 - f(c_k)$ for $k = p+3, p+4, ..., 2p$, also contradicting $e^O(0) = 2$.

Case (c): (c_1,c_p) and (c_p,c_{2p}) are two inner 0-edges. In this case, $f(c_1) = 1 - f(c_i)$ for $i = 2, 3, 4, ..., p - 1$, also contradicting $e^O(0) = 2$.

If $e^O(0) = 4$, then $e^I(0) = 0$. Suppose $f(c_1) = x$, then $f(c_i) = 1 - x$ for $i = 3, 4, ..., p, p+1$. This implies that $f(c_j) = x$ for $j = p+3, p+4, ..., 2p$. Since $p \geq 6$, this implies that $e^O(0) \geq 8$, a contradiction.

This completes the proof. □

3. Two Classes of Maximal Outerplanar Graphs:

Class 1. $M_1(n)$, $n \geq 4$

$V(M_1(n)) = \{c_1, c_2, c_3, ..., c_{n-1}, c_n\}$

$E(M_1(n)) = E(C_n) \cup \{(c_1,c_3), (c_1,c_4), (c_1,c_5), ..., (c_1,c_{n-2}), (c_1,c_{n-1})\}$

Lemma 3.1:

If n is even, then $\max \{FI(M_1(n))\} = n - 1$.

Proof:

Let n = 2p. The result is trivial if n = 4. Let n ≥ 6. We first show that max $\{FI(M_1(n))\} \le n - 1$. To show this, we seek to minimize e(0) or e(1). For the standard labeling, e(0) = p – 1 giving e(1) – e(0) = 2p – 1 = n – 1. It suffices to show that min {e(0)} or min {e(1)} ≥ p – 1. By Lemma 1.3, we do not need to consider min {e(1)} if $e^I(1) \ge p - 3$ as this implies min {e(1)} ≥ p – 1.

First assume $f(c_1) = f(c_2) = f(c_n) = x$ (x = 0 or 1). Since the labeling is friendly, we have $e^I(0) = p - 3$ and $e^I(1) = p$. Now, min $\{e^O(0)\} = 4$ is attained if we label c_3 to c_{2p-2} with 1 – x and x alternately, and label c_{2p-3} to c_{2p-1} with 1 – x. Thus, min {e(0)} = p + 1 > p – 1.

Now assume $f(c_1) = f(c_2) = x$ and $f(c_n) = 1 - x$ (x = 0 or 1). Since the labeling is friendly, we have $e^I(0) = p - 2$ and $e^I(1) = p - 1$. Now, min $\{e^O(0)\} = 2$ is attained if we label c_3 to c_{2p-1} with 1 – x and x alternately. Thus, min {e(0)} = p > p – 1.

Finally, assume $f(c_1) = x$ and $f(c_2) = f(c_n) = 1 - x$ (x = 0 or 1). Since the labeling is friendly, we have $e^I(0) = p - 1$ and $e^I(1) = p - 2$. Now, min $\{e^O(0)\} = 0$ is attained if we label c_3 to c_{2p-1} with x and 1 – x alternately. Thus, min {e(0)} = p – 1. Therefore, max $\{FI(M_1(n))\} = 3p - 2 - (p - 1) = n - 1$. □

Theorem 3.1:

If n is even, $FI(M_1(n)) = \{1,3, \dots, n-1\}$.

Proof:

Let n = 2p. By Theorem 1.1 and Lemma 3.1, it suffices to show that the values in {1, 3, …, n – 1} are attainable. Note that c_1 is the maximum degree vertex. For each i = 0, 1, …, p – 1, let $f(c_1) = f(c_3) = \dots = f(c_{2i+1}) = 1$, $f(c_2) = f(c_4) = \dots = f(c_{2i+2}) = 0$, $f(c_{2i+3}) = f(c_{2i+4}) = \dots = f(c_{i+p+1}) = 0$, $f(c_{i+p+2}) = f(c_{i+p+3}) = \dots = f(c_{2p}) = 1$. Then v(0) = v(1), i.e., the vertex labeling is friendly. Observe that the values $f(c_1) = 1$, $e^I(0) = p - 2$ and $e^I(1) = p - 1$ for all possible i. Counting the outer edge labels, we have that $e^O(0) = 2p - 2i - 2$, and $e^O(1) = 2i + 2$, with $e^O(0) - e^O(1) = 2p - 4i - 4$ and $e^I(0) - e^I(0) = -1$. Letting i from 0 to p – 2, we have that e(0) – e(1) = 2p – 4i – 5 = {2p – 5, 2p – 9, 2p – 13, …, 7, 3, – 1, – 5, … , – 2p + 11, – 2p + 7, – 2p + 3} if p is even, and e(0) – e(1) = {2p – 5, 2p – 9, 2p – 13, …, 9, 5, 1, – 3, … , – 2p + 11, – 2p + 7, – 2p + 3} if p is odd. Thus, $FI(M_1(n))$ = {1, 3, 5, …, 2p – 7, 2p – 5, 2p – 3} with 2p – 1 missing. When i = p – 1, it can be checked directly that $e^O(0) = 0$ and $e^O(1) = 2p$ while $e^I(0) = p - 1$ and $e^I(1) = p - 2$. This gives us e(0) – e(1) = – 2p + 1. This completes the proof. □

Lemma 3.2:

If n is odd, then max $\{FI(M_1(n))\} = n - 2$.

Proof:

Let n = 2p + 1. For the standard labeling, e(0) = p giving e(1) – e(0) = 2p + 1 = n – 2. It suffices to show that min {e(0)} or min {e(1)} ≥ p. By Lemma 1.3, we do not need to consider min {e(1)} if $e^I(1) \ge p - 2$ as this implies min {e(1)} ≥ p. By an argument similar to that in the proof of Theorem 2.1, we can get the values of $e^I(0)$, $e^I(1)$ and min {e(0)} for all possible different friendly labelings as in Table 3.1.

Table 3.1: Values of min {e(0)} and min {e(1)} for $M_1(n)$, n = 2p + 1.

Cases (x = 0 or 1)	|v(x)|	$e^I(0)$	$e^I(1)$	min$\{e^O(0)\}$	min{e(0)}
$F(c_1) = f(c_2) = f(c_n) = x$	P	p – 3	p + 1	5	p + 2
	p + 1	p – 2	p	3	p + 1
$f(c_1) = f(c_2) = x$, $f(c_n) = 1 - x$	P	p – 2	p	3	p + 1
	p + 1	p – 1	p – 1	1	p
$f(c_1) = x$, $f(c_2) = f(c_n) = 1 - x$	P	p – 1	p – 1	1	p
	p + 1	p	p – 2	1	p + 1

From Table 3.1, it is clear that max $\{FI(M_1(n))\} = 2p - 1 = n - 2$. □

Theorem 3.2:

If n is odd, $FI(M_1(n)) = \{1,3, \dots, n - 2\}$.

Proof:

Let n = 2p+1. By Theorem 1.1 and Lemma 3.2, it suffices to show that the values in {1, 3, …, n – 2} are attainable. For each i = 0, 1, …, p – 1, let $f(c_1) = f(c_3) = \dots = f(c_{2i+1}) = 1$, $f(c_2) = f(c_4) = \dots = f(c_{2i+2}) = 0$, $f(c_{2i+3}) = f(c_{2i+4}) = \dots = f(c_{i+p+1}) = 0$, $f(c_{i+p+2}) = \dots = f(c_{2p}) = f(c_{2p+1}) = 1$. Then v(1) = v(0) + 1, i.e., the vertex labeling is friendly. Observe that the values $f(c_1) = 1$, $e^I(0) = p - 1 = e^I(1)$ for all possible i. Counting the outer edge labels, we have that $e^O(0) = 2p - 2i - 1$, and $e^O(1) = 2i + 2$, with $e^O(0) - e^O(1) = 2p - 4i - 3$. Letting i from 0 to $p - 1$, we have that $e(0) - e(1) = \{2p - 3, 2p - 7, \ldots, 5, 1, -3, -7, \ldots, -2p + 5, -2p + 1\}$ if p is even, and $e(0) - e(1) = \{2p - 3, 2p - 7, \ldots, 3, -1, -5, \ldots, -2p + 5, -2p + 1\}$ if p is odd. Thus, $FI(M_1(n)) = \{1, 3, 5, \ldots, 2p - 3, 2p - 1\}$. This completes the proof. □

Example 3:

$FI(M_1(4)) = FI(M_1(5)) = \{1,3\}$.

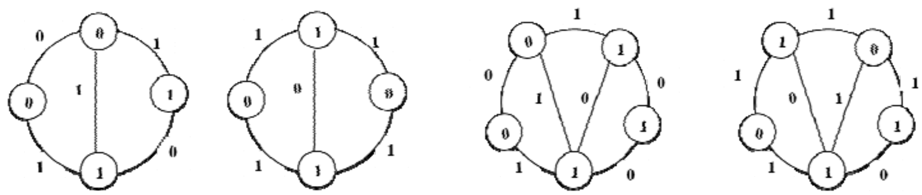

Fig. 3: Some friendly labeling of $M_1(4)$ and $M_1(5)$.

Example 4:

$FI(M_1(6)) = FI(M_1(7)) = \{1,3,5\}$.

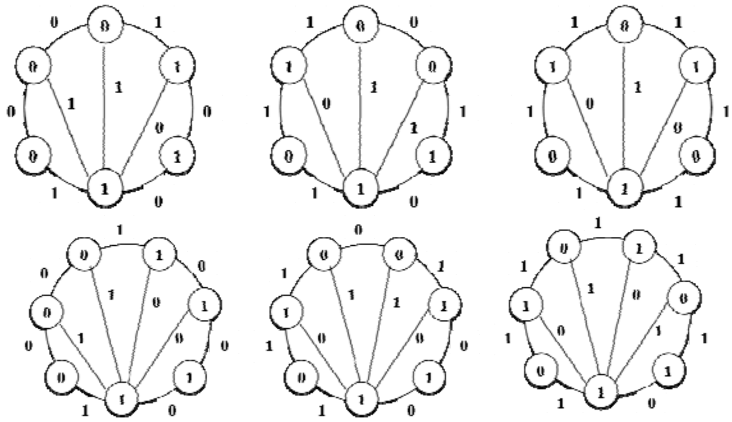

Fig. 4: Some friendly labeling of $M_1(6)$ and $M_1(7)$.

Example 5:

$FI(M_1(8)) = FI(M_1(9)) = \{1,3,5,7\}$.

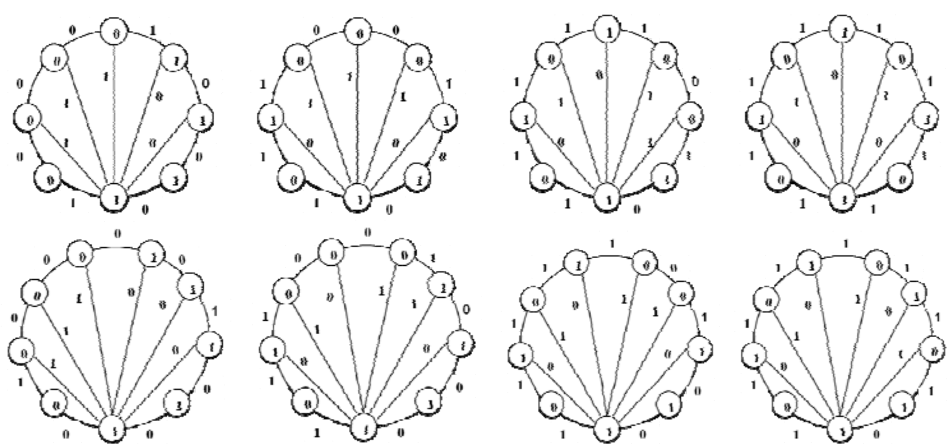

Fig. 5: Some friendly labeling of $M_1(8)$ and $M_1(9)$.

Class 2. $M_2(n)$. For $n \geq 6$.:

$V(M_2(n)) = \{c_1, c_2, c_3, ..., c_{n-1}, c_n\}$

$E(M_2(n)) = E(C_n) \cup \{(c_1, c_3), (c_1, c_5), (c_1, c_6), ..., (c_1, c_{n-2}), (c_1, c_{n-1})\} \cup \{(c_3, c_5)\}$

Lemma 3.3:

If n is even, $\max \{FI(M_2(n))\} = n - 3$.

Proof:

Let n = 2p. For the standard labeling with $f(c_i) = x$ for all odd i, e(0) = p gives e(1) – e(0) = 2p – 3 = n – 3. It suffices to show that min {e(0)} or min {e(1)} ≥ p. By Lemma 1.3, we do not need to consider min {e(1)} if $e^I(1) \geq p - 2$ as this implies min {e(1)} ≥ p. We consider four cases: (a) $f(c_1) = f(c_2) = f(c_n) = x$ (x = 0 or 1), (b) $f(c_1) = f(c_2) = x$, $f(c_n) = 1 - x$, (c) $f(c_1) = f(c_n) = x$, $f(c_2) = 1 - x$, and (d) $f(c_1) = x$, $f(c_2) = f(c_n) = 1 - x$. We get the values of $e^I(0)$, $e^I(1)$, $\min\{e^O(1)\}$ (if applicable) and min {e(0)} for all possible different friendly labelings as in Table 3.2.

Table 3.2: Values of min {e(0)} and min {e(1)} for $M_2(n)$, n = 2p.

Case (a)	$f(c_4)$	$e^I(0)$	$e^I(1)$	$\min\{e^O(0)\}$	$\min\{e(0)\}$
$f(c_3) = f(c_5) = x$	x	p – 3	p	10	p + 7
	1 – x	p – 2	p – 1	6	p + 4
$f(c_3) = f(c_5) = 1 - x$	x	p – 3	p	4	p + 1
	1 – x	p – 2	p – 1	4	p + 2
$f(c_3) = x$, $f(c_5) = 1 - x$	x	p – 4	p + 1	8	p + 4
	1 – x	p – 3	p	6	p + 3
$f(c_3) = 1 - x$, $f(c_5) = x$	x	p – 4	p + 1	6	p + 2
	1 – x	p – 3	p	4	p + 1

Case (b)	$f(c_4)$	$e^I(0)$	$e^I(1)$	$\min\{e^O(0)\}$	$\min\{e(0)\}$
$f(c_3) = f(c_5) = x$	x	p – 2	p – 1	8	p + 6
	1 – x	p – 1	p – 2	4	p + 3
$f(c_3) = f(c_5) = 1 - x$	x	p – 2	p – 1	2	p
	1 – x	p – 1	p – 2	4	p + 3
$f(c_3) = x$, $f(c_5) = 1 - x$	x	p – 3	p	6	p + 3
	1 – x	p – 2	p – 1	4	p + 2
$f(c_3) = 1 - x$, $f(c_5) = x$	x	p – 3	p	4	p + 1
	1 – x	p – 2	p – 1	2	p

Case (c)	$f(c_4)$	$e^I(0)$	$e^I(1)$	$\min\{e^O(0)\}$	$\min\{e(0)\}$
$f(c_3) = f(c_5) = x$	x	p – 2	p – 1	6	p + 4
	1 – x	p – 1	p – 2	2	p + 1
$f(c_3) = f(c_5) = 1 - x$	x	p – 2	p – 1	2	p
	1 – x	p – 1	p – 2	6	p + 5
$f(c_3) = x$, $f(c_5) = 1 - x$	x	p – 3	p	4	p + 1
	1 – x	p – 2	p – 1	2	p
$f(c_3) = 1 - x$, $f(c_5) = x$	x	p – 3	p	4	p + 1
	1 – x	p – 2	p – 1	4	p + 2

Case (d)	$f(c_4)$	$e^I(0)$	$e^I(1)$	$\min\{e^O(0)\}$	$\min\{e^O(1)\}$	$\min\{e(0)\}$
$f(c_3) = f(c_5) = x$	x	p – 1	p – 2	4	--	p + 3
	1 – x	p	p – 3	0	6	p
$f(c_3) = f(c_5) =$	x	p – 1	p – 2	2	--	p + 1
$1 - x$	1 – x	p	p – 3	6	4	p + 6
$f(c_3) = x$,	x	p – 2	p – 1	2	--	p
$f(c_5) = 1 - x$	1 – x	p – 1	p – 2	2	--	p + 1
$f(c_3) = 1 - x$,	x	p – 2	p – 1	2	--	p
$f(c_5) = x$	1 – x	p – 1	p – 2	4	--	p + 3

From Table 3.2, it is clear that $\max \{FI(M_2(n))\} = 2p - 3 = n - 3$. □

Theorem 3.3:

If n is even, $FI(M_2(n)) = \{1,3, ,…, n - 3\}$.

Proof:

Let n = 2p. By Theorem 1.1 and Lemma 3.3, it suffices to show that the values in {1, 3, …, n – 3} are attainable. We consider two cases.

Case 1. n ≡ 0 (mod 4). Begin with the standard labeling with e(0) = p and e(1) = 3p – 3. Divide the vertices c_5 to c_n into (n – 4)/4 blocks of 4 vertices, each with vertex labels 0, 1, 0, 1. Start with the block with c_5 to c_8 and change the vertex labels to 0,0,1,1. Then e(0) increases by 2 and edge (c_3,c_5) still has label 0. Continue with the rest of the blocks of 4 vertices and e(0) finally reaches p + p – 2 = 2p – 2 in increments of 2, and e(1) decreases to 2p – 1. Hence, e(1) – e(0) changes from 2p – 3 = n – 3 to 1 in decrements of 4.

Now, begin with the standard labeling again and change the labels of c_7 to c_n to their complements so that edge (c_3,c_5) still has label 0. This labeling is friendly with v(0) = v(1), e(0) = p + 1 and e(1) = 3p – 4. If n ≥ 12, divide the vertices c_9 to c_n into (n – 8)/4 blocks of 4 vertices, each with vertex labels 1, 0, 1, 0. Start with the block with c_9 to c_{12} and change the vertex labels to 1, 1, 0, 0. Then e(0) increases by 2. Continue with the rest of the blocks of 4 vertices. We see that e(0) finally reaches p + 1 + p – 4 = 2p – 3 in increments of 2 from p + 1, and e(1) decreases to 2p + 2. Hence, e(1) – e(0) changes from 2p – 5 = n – 5 to 3 in decrements of 4.

Thus, FI(M_3(n)) = {1, 3, 5, …, n – 3}.

Case 2. n ≡ 2 (mod 4). If n = 6, label vertices c_1, c_2, …, c_6 by 0, 0, 0, 1, 1, 1 and 0, 1, 0, 1, 0, 1 to get e(1) – e(0) = 1 and 3, respectively. Now assume n ≥ 10. Begin with the standard labeling with e(0) = p and e(1) = 3p – 3. Divide the vertices c_5 to c_{n-2} into (n – 6)/4 blocks of 4 vertices, each with vertex labels 0, 1, 0, 1. Start with the block with c_5 to c_8 and change the vertex labels to 0, 1, 1, 0. Then e(0) increases by 2. Continue with the rest of the blocks, and e(0) finally reaches p + p – 3 = 2p – 3 in increments of 2, and e(1) decreases to 2p. Hence, e(1) – e(0) changes from 2p – 3 = n – 3 to 3 in decrements of 4.

Now, begin with the standard labeling again and change the labels of c_7 to c_n to their complements so that edge (c_3,c_5) still has label 0. The labeling is friendly with v(0) = v(1) and e(0) = p + 1 and e(1) = 3p – 4. Divide the vertices c_7 to c_n into (n – 6)/4 blocks of 4 vertices, each with vertex labels 1, 0, 1, 0. Start with the block with c_7 to c_{10} and change the vertex labels to 1, 1, 0, 0. Continue with the rest of the blocks of 4 vertices. We see that e(0) finally reaches p + 1 + p – 3 = 2p – 2 in increments of 2 from p + 1, and e(1) decreases to 2p – 1. Hence, e(1) – e(0) changes from 2p – 5 = n – 5 to 1 in decrements of 4.

Thus, FI(M_3(n)) = {1, 3, 5, …, n – 3}. □

Theorem 3.4:

If n is odd, FI(M_2(n)) = {1, 3, 5,…, n – 2}.

Proof:

Let n = 2p+1. By an argument similar to that in Lemma 3.3, we need to consider |v(x)| = p or p + 1 in each Case (a) to Case (d). Thus, there are 16 subcases each. The readers can verify that max {FI(M_2(n))} = n – 2 and it is attained under our standard labeling. By Theorem 1.1, it suffices to show that the values in {1, 3, …, n – 3} are attainable. We consider two cases.

Case 1: n ≡ 1 (mod 4). Begin with the standard labeling. Change the vertex labels of c_3 to c_n to their complements to get e(0) = p and e(1) = 3p – 1. Divide the vertices c_5 to c_{2p} into (2p – 4)/4 blocks of 4 vertices, each with vertex labels 1, 0, 1, 0. Start with the block with c_5 to c_8 and change the vertex labels to 1, 1, 0, 0. Then e(0) increases by 2 and edge (c_3,c_5) still has label 0. Continue with the rest of the blocks, and e(0) finally reaches p + p – 2 = 2p – 2 in increments of 2, and e(1) decreases to 2p + 1. Hence, e(1) – e(0) changes from 2p – 1 = n – 2 to 3 in decrements of 4.

Now, begin with the standard labeling with e(0) = p + 1 and e(1) = 3p – 2. Divide the vertices c_5 to c_{2p} into (2p – 4)/4 blocks of 4 vertices, each with vertex labels 0, 1, 0, 1. Start with the block with c_5 to c_8 and change the vertex labels to 0, 0, 1, 1. Then e(0) increases by 2. Continue with the rest of the blocks of 4 vertices. We see that e(0) finally reaches p + 1 + p – 2 = 2p – 1 in increments of 2 from p + 1, and e(1) decreases to 2p. Hence, e(1) – e(0) changes from 2p – 3 = n – 4 to 1 in decrements of 4.

Thus, FI(M_3(n)) = {1, 3, 5, …, n – 2}.

Case 2: n ≡ 3 (mod 4). Begin with the standard labeling. Change the vertex labels of c_3 to c_n to their complements to get e(0) = p and e(1) = 3p – 1. Divide the vertices c_5 to c_{2p-2} into (2p – 6)/4 blocks of 4 vertices, each with vertex labels 1, 0, 1, 0. Start with the block with c_5 to c_8 and change the vertex labels to 1, 1, 0, 0. Then e(0) increases by 2 and edge (c_3,c_5) still has label 0. Continue with the rest of the blocks, and e(0) finally reaches p + p – 3 = 2p – 3 in increments of 2, and e(1) decreases to 2p + 2. Hence, e(1) – e(0) changes from 2p – 1 = n – 2 to 5 in decrements of 4. Finally, change the vertex label of v_n (i.e. 1) to its complement to get e(1) – e(0) = 2p – (2p – 1) = 1.

Now, begin with the standard labeling with e(0) = p + 1 and e(1) = 3p – 2. Divide the vertices c_5 to c_{2p} into (2p – 4)/4 blocks of 4 vertices, each with vertex labels 0, 1, 0, 1. Start with the block with c_5 to c_8 and change the vertex labels to 0, 1, 1, 0. Then e(0) increases by 2. Continue with the rest of the blocks of 4 vertices. We see that e(0) finally reaches p + 1 + p – 3 = 2p – 2 in increments of 2 from p + 1, and e(1) decreases to 2p + 1. Hence, e(1) – e(0) changes from 2p – 3 = n – 4 to 3 in decrements of 4.

Thus, FI(M_3(n)) = {1, 3, 5, …, n – 2}. □

Example 6:

FI(M_2(6)) = {1, 3}, FI(M_2(7)) = FI(M_2(8)) = {1,3,5}.

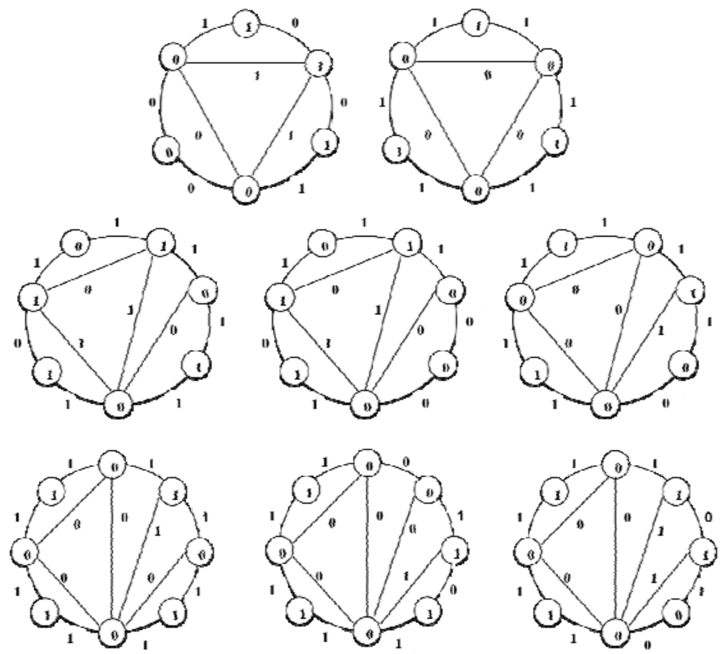

Fig. 6: Some friendly labeling of $M_2(7)$ and $M_2(8)$.

Example 7:

$FI(M_2(9)) = FI(M_2(10)) = \{1, 3, 5, 7\}$.

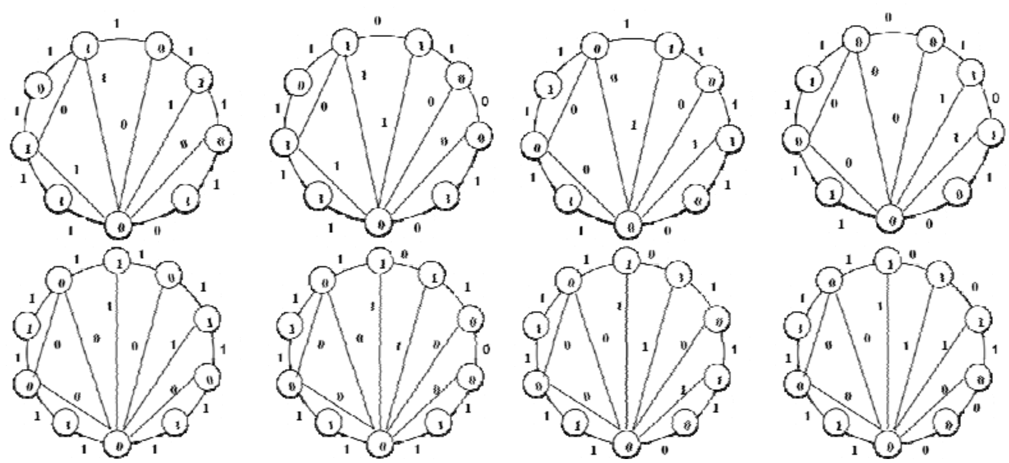

Fig. 7: Some friendly labeling of $M_2(9)$ and $M_2(10)$.

4. Conclusion & Discussion:

Our Theorem 2.3 shows that not all maximal outerplanar graphs have friendly index set that form an arithmetic progression. Sallehi and De (2009) also showed that not all trees have friendly index set that form an arithmetic progression. Lee and Ng (2008) also show that the friendly index sets of complete bipartite graphs do not form an arithmetic progression.

Corollary 4.1:

There exists both planar and non-planar graphs with friendly index sets not forming an arithmetic progression.

Conjecture 4.1:

There are infinitely many family of graphs whose friendly index sets not forming an arithmetic progression.

The proof of Theorem 2.2 implies that there are infinitely many families of MOP with max{FI} = 2n – 9 and min {e(0)} > min {e(1)} = 3. By Theorems 2.1, 2.2, 3.1 and 3.2, there also exists MOPs of order n = 6, 7, 8 that admit a friendly labeling with max{FI} = 2n – 9 and min {e(1)} > min {e(0)} = 3. However, we are not able to show that this is true for n ≥ 9.

Conjecture 4.2:

Let G be a MOP of order n ≥ 9, then max {FI(G)} = 2n – 9 if and only if min{e(0)} > min{e(1)} = 3.

Problem 4.1:

Characterize all maximal outerplanar graphs G of order n with FI(G) ⊂ {1, 3, 5, ..., 2n – 13, 2n – 9} .

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