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An Inventory Model with Change in Demand Distribution

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Abstract: In this work, a modified version of the inventory model is discussed under the assumption that the random variable denoting demand undergoes a change in the distribution after a particular value of 'X', denoted as x_0, which is called change point or truncation point. In the very simple case it is assumed that the random variable X follows exponential distribution with parameter θ_1 prior to the truncation point x_o and it follows Erlang2 distribution with parameter θ_2 after the truncation point. In this paper, it is assumed that the distribution of this random variable undergoes a change after a certain value of the random variable X, called the change point. Using this concept the optimal value of one time supply is determined and suitable numerical illustration is also furnished.

Key words: Inventory control, SCBZ Property, Change point, Demand Distribution.

INTRODUCTION

A major part of the literature in this branch was developed by people like Arrow K.J *et al.* (1958) Application of dynamic programming principles to solve production smoothing problems was carried out by Bellman.R (1957). Little J.D.C (1955) has developed an inventory model for the optimal discharge of water from dam for the purpose of production of electricity. Introducing the concept of dynamic behavior of variables like demand, supply and also the cost of Economic Order Quantity model under dynamic situation was developed by Arrow K.J *et al*. (1951). Srinivasan (2006**)** have discussed an inventory model in which the demand over the time interval [0, t] is denoted as a random variable with the probability density function *f*(x). The newsboy problem is another interesting inventory model. The inventory or stock can be kept only for a finite duration after which the product has got no value or it has a negligible value. So, the model involves the concept of finite process. The so-called newsboy problem is one in which the product is taken to be a newspaper and it should be sold on the same day otherwise it can have only a negligible price as waste paper. The determination of optimal supply size has been attempted by Bernatt.H (1961) and he developed a similar model. Many other variations of this model have been considered by Levi (1960) and Hadley *et al.***(**1961).

In inventory control theory, there are some situations where the optimal one time supply during the interval [0, t] is considered, taking into consideration of the holding cost and shortage cost. The demand is taken to be a random variable in the so-called base stock system for patient customers. A new type of inventory mechanism has been developed by Gaver (1959) which is called the base stock inventory model. In real life situation the demand is a random variable in almost all cases. In inventory management the determination of the optimal size of the inventory is of prime interest, since it minimizes the overall total cost arising due to the holding cost of excess inventory and also shortage cost due to inadequate stock. There are many situations where the one time supply during the period [0, t] is considered along with the demand for the product in the same period. The demand is taken to be a random variable and it is not a control variable. But the supply of the product, which in other words the order quantity, is under the control of decision maker. The problem of interest here is the determination of one time optimal supply during a specified time interval [0, t]. In this model the expected total cost is given as

$$E(C) = h\int_0^s (S-X)f(x)dx + d\int_s^\infty (X-S)f(x)dx \tag{1}$$

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Where h- Holding cost
d- Shortage cost
S- Supply size or initial stock level
X- Demand
The optimal value of the supply denoted as S can be obtained by using the result

$$\frac{dE(C)}{dS}=0$$

Since it is the differentiation of an integral with respect to the variable S, which is both in the integrand, as well as in the limits of integration, we have to use the formula indicated in

$$\frac{d}{dx}\int_{\varphi(x)}^{\psi(x)} f(x,t)\,dt=\psi'(x)\,f(x,\psi(x))-\varphi'(x)\,f(x,\varphi(x))+\int_{\varphi(x)}^{\psi(x)}\frac{d}{dx}f(x,t)dt$$

(2) Using this result it can be shown that $F(\hat{S})=\dfrac{d}{d+h}$

which implies that the optimal value of S satisfies the equation

$$\int_0^s f(x)dx=F\left(\hat{S}\right)=P\left[X<s\right]=\frac{d}{d+h}$$

Given the values of the inventory holding cost 'h' and shortage cost 'd' and the probability distribution f(x) of the random variable 'X' denoting demand and optimal 'S' can be determined. This basic model discussed by Hannsman (1962).

In this work a modified version of the model is discussed under the assumption that the random variable denoting demand undergoes a change in the distribution after a particular value of 'X', denoted as x_0, which is called change point or truncation point. In the very simple case it is assumed that the random variable X follows exponential distribution with parameter θ_1 prior to the truncation point x_o and it follows Erlang2 distribution with parameter θ_2 after the truncation point. It may be observed that the exponential distribution possesses the so called *Lack of Memory Property* (LMP) where as Erlang2 distribution does not satisfy LMP. Using this concept the optimal value of S is derived under two cases namely,

i) x_0 is fixed
ii) x_0 itself is a random variable.

The concept of SCBZ property indicates that a random variable X with density function *f(x)* undergoes a parametric change after a particular value of X denoted as x_0. This is a slight modification of the Lack of Memory property. A slight modification of the property has been suggested by Raja Rao and Talwaker (1990). An extension of this concept leads to the concept of change of distribution after a change point.

For example, if X is a random variable denoting the life time of the component and *f(x, θ)* is the probability density function, we say that a random variable undergoes a change of distribution after a change point when the following condition is satisfied.

X has the probability density function f(x) and cumulative distribution function F(X) whenever $X \le x_0$ and it has probability density function h(x) with cumulative distribution function H(X) whenever $X > x_0$. x_0 is called the change point. It can be noted that $\int_o^{x_o} f(x)dx+\int_{x_o}^{\infty} h(x)dx=1$

This property was initially introduced by Stangl.D.K (1995). An application of this property in shock model cumulative damage process has been introduced by Suresh Kumar.R (2006).

The use of change of distribution after a change point is justified by the fact that demand for any product over the time interval [0, t] is not fixed. A Fluctuation in demand is a very common phenomenon. If the demand for the product is accordingly to some probability distribution initially, it is very likely that after a certain point, the demand may undergo some changes and the rate of increase in demand (or) the rate of decrease in demand will undergo considerable change. Hence, to depict the demand as random variable undergoing a change of distribution after the particular magnitude x_0 is quite reasonable.

2. Assumptions of the Model:

i) The demand for the product is a random variable and probability distribution of the random variable undergoes a change in its distribution after a change point.
ii) The change point or the truncation point is either a constant or itself can be a random variable which is exponentially distributed with parameter λ.
iii) There is a onetime supply to meet the demand over the period [0, t].
iv) The initial inventory starts with the supply of size S.

Notations:

X= a random variable denoting the demand and it has probability density function
f(x) and cumulative distribution function F(x).
h = Inventory holding cost /unit of product
d = Shortage cost/unit
S = the supply size or initial stock level
x_0= A value of X which is the truncation point or change point
$\hat{S}$ = optimal value of S

4. Models:

4.1 Model 1:

Here x_0 is constant.
In the present model it is assumed that the demand function f(x) is such that

$f(x) = f_1(x)$ if $X \le x_0$
$\quad = f_2(x)$ if $X > x_0$
We also have $f_1(x) \sim \exp(\theta_1)$
$f_2(x) \sim$ Erlang2 (θ_2)

It may be seen that $f(x) = f_1(x)$ if $X \le x_0$
$= f_2(x)$ if $X > X_0$

$$f_1(x) = \theta_1 . e^{-\theta_1 x}$$

and $$f_2(x) = \overline{F_1(x_0)}\, f_2(x - x_0) \quad \text{if } X > X_0$$

where $\overline{F_1}(x_0) = 1 - F_1(x_0)$. This result has been proved by Suresh Kumar.R (2006)

Since $f_1(x) \sim \exp(\theta_1)$ and

$f_1(x - x_0)$ ~ Erlang2 with parameter θ_2

We have $f_1(x) = \theta_1 . e^{-\theta_1 x}$

$$f_2(x) = \theta_2^2 (x - x_0) e^{-\theta_2 (x - x_0)} . e^{-\theta_1 x_0}$$

since $\overline{F_1}(x_0) = 1 - F_1(x_0) = e^{-\theta_1 x_0}$

Using this result the expression for expected cost can be written as:

4.2. Model 1 a): $x_0 > s$

$$E(C)=h.\int_0^s (S-X)f_1(x)dx + d\int_s^{x_0}(X-S)f_1(x)dx + d\int_{x_0}^{\infty}(X-S)f_2(x)dx \quad (3)$$

$$E(C) = I_1 + I_2 + I_3$$

Now differentiating E(C) with respect to S we have,

$$\frac{dI_1}{ds}=h.\int_0^s (S-X)f_1(x)dx = h\theta_1\int_0^s e^{-\theta_1 x}dx = h\left\{1-e^{-\theta_1 s}\right\} \quad (4)$$

$$\frac{dI_2}{ds}=d\int_s^{x_0}(X-S)f_2(x)dx = -d\theta_1\int_s^{x_0} e^{-\theta_1 x}dx = d\left\{e^{-\theta_1 x_0}-e^{-\theta_1 s}\right\} \quad (5)$$

$$\frac{dI_3}{ds}=\int_{x_0}^{\infty}(X-S)f_2(x)dx = \int_{x_0}^{\infty}(X-S)\theta_2^2(x-x_0)e^{-\theta\,(x-x_0)}e^{-\theta_1 x_0}dx = -de^{-\theta_1 x_0} \quad (6)$$

hence using (4), (5) and (6) gives

$$h\left\{1-e^{-\theta_1 s}\right\}+d\left\{e^{-\theta_1 x_0}-e^{-\theta_1 s}\right\}-de^{-\theta_1 x_0}=0$$

$$h\left\{1-e^{-\theta_1 s}\right\}=de^{-\theta_1 s}$$

$$e^{-\theta_1 s}=\frac{h}{d+h} \quad (7)$$

Any value of S, which satisfies (7), is the optimal S.

4.2.1 Numerical Illustrations:

Case i)

θ_1 = 1.5; h = 5

d	10	20	30	40	50
S	0.7324	1.0729	1.2972	1.4648	1.5985

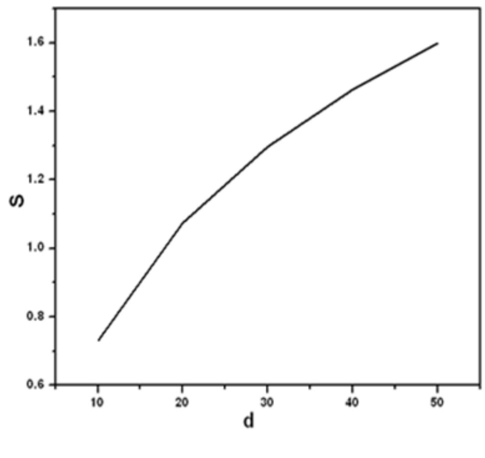

Fig. 1:

Case ii)

θ_1 = 1.5; d = 10

H	5	10	15	20	25
S	0.7324	0.4620	0.3405	0.2703	0.2243

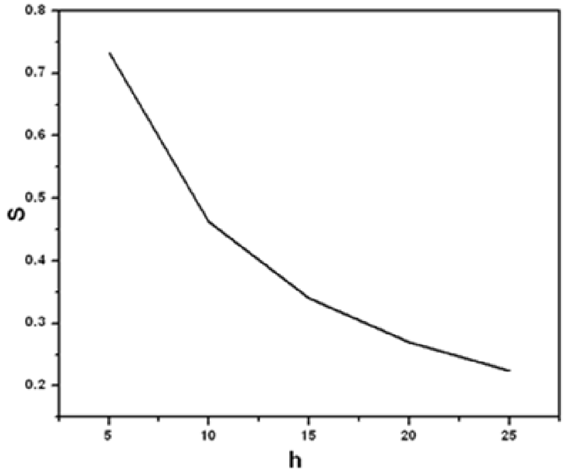

Fig. 2:

Inferences:

It is seen from *Case i* that, as the value of the shortage cost d increases the optimal supply size also increases. This is quite reasonable because the shortage cost is increasing and hence keeping higher level of stock must minimize the level of shortage.

It is seen from *Case ii* that, as the inventory holding cost increases the optimal reserve size decreases. It is also justified by the fact that if the inventory holding cost is higher, then it is reasonable to have a smaller supply size.

Model 1b): Here $x_0 < S$

$$E(C)=h\int_0^{x_0}(S-X)f_1(x)dx+h\int_{x_0}^{s}(S-X)f_2(x)dx+d\int_s^{\infty}(X-S)f_2(x)dx \tag{8}$$

gives

$$e^{-\theta_2 s}\left[\theta_2 s-\theta_2 x_0+1\right]=\frac{h}{(h+d).e^{x_0(\theta_2-\theta_1)}} \tag{9}$$

Any value of S which satisfies (9) is the optimal value ‘S’.

Numerical Illustrations:

Case i

θ_1 = 1.0; θ_2 =2.0; x_0=20; d=10 all are fixed

H	5	10	15	20	25	30
S	4.480	4.305	4.227	4.182	4.152	4.131

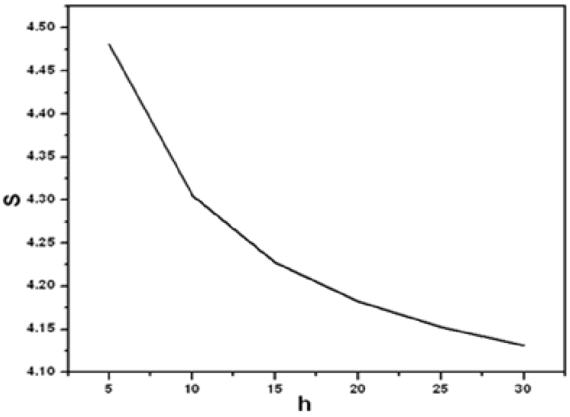

Fig. 3:

Case ii)

θ_1 =1.0; θ_2=2.0; x_0=20; h=5 all are fixed

D	10	20	30	40	50
S	4.4804	4.7012	4.8467	4.9555	5.0424

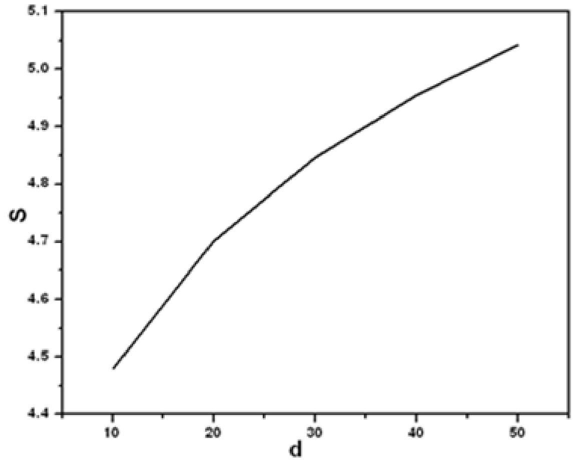

Fig. 4:

Inferences:

In *Case i*, as the inventory holding cost 'h' increases, it is desirable to have smaller inventory.

In *Case ii*, it is well known fact that larger inventory should be suggested as the value of shortage cost increases. It is observed that the same kind of suggestion is applicable to the present also.

Model 2: x_0 is a random variable

x_0 is random variable which follows exponential with parameter λ

when X < x_0 $\quad f_1(x) = \theta_1.e^{-\theta_1 x}.p\left[x < x_0\right]$

$$= \theta_1.e^{-\theta_1 x}.p\left[x_0 > x\right]$$

$$= \theta_1.e^{-\theta_1 x}.e^{-\lambda x}$$

since $x \sim \exp(\lambda)$ and $P\left[x_0 > x\right] = e^{-\lambda x}$

$$f_1(x) = \theta_1 e^{-x(\theta_1 + \lambda)}$$

when X > x_0

$$f_2(x)=\theta_2^2(x-x_0)\,e^{-\theta_2(x-x_0)}\,.e^{-\theta_1 x_0}.P\left[x>x_0\right]$$

$$=\theta_2^2(x-x_0)\,e^{-\theta_2(x-x_0)}\,.e^{-\theta_1 x_0}.P\left[x_0<x\right]$$

$$=\theta_2^2(x-x_0)\,e^{-\theta_2(x-x_0)}\,.e^{-\theta_1 x_0}.\lambda.e^{-\lambda x_0}dx$$

$$f_2(x)=\lambda\theta_2^2\left[\frac{e^{-x(\theta_1+\lambda)}}{(\theta_1-\theta_2+\lambda)^2}\right]-\left[\frac{e^{-x\theta_2}}{(\theta_1-\theta_2+\lambda)^2}\right]+\left[\frac{x.e^{-x\theta_2}}{(\theta_1-\theta_2+\lambda)}\right]$$

Model 2a: x_0 > S

$$E(C)=h\int_0^S(S-X)f_1(x)dx+d\int_S^{x_0}(X-S)f_1(x)dx+d\int_{x_0}^{\infty}(X-S)f_2(x)dx \quad (10)$$

$E(C)=I_1+I_2+I_3$

Now differentiating E(C) with respect to S and equating it to zero, we have

$\frac{dE(C)}{dS}=0$

$$\frac{dI_1}{dS}=h\int_0^S(S-X)f_1(x)dx\ =h\int_0^S\theta_1 e^{-x(\theta_1+\lambda)}dx\ =\frac{h\theta_1}{\theta_1+\lambda}\left\{\left[1-e^{-S(\theta_1+\lambda)}\right]\right\} \quad (11)$$

$$\frac{dI_2}{dS}=d\int_S^{x_0}(X-S)f_1(x)dx\ =-d\int_S^{x_0}f_1(x)dx\ =-d\int_S^{x_0}\theta_1 e^{-x(\theta_1+\lambda)}dx\ =d\theta_1\left\{\left[\frac{.e^{-x_0(\theta_1+\lambda)}}{\theta_1+\lambda}\right]-\left[\frac{.e^{-S(\theta_1+\lambda)}}{(\theta_1+\lambda)}\right]\right\} \quad (12)$$

$$\frac{dI_3}{dS}=d\int_{x_0}^{\infty}(X-S)f_2(x)dx\ =d\lambda\theta_2^2\int_{x_0}^{\infty}(X-S)\left[\frac{e^{-x(\theta_1+\lambda)}}{(\theta_1-\theta_2+\lambda)^2}\right]-\left[\frac{e^{-x\theta_2}}{(\theta_1-\theta_2+\lambda)^2}\right]+\left[\frac{x.e^{-x\theta_2}}{(\theta_1-\theta_2+\lambda)}\right]dx$$

$$=d\lambda\theta_2^2\left\{\begin{array}{l}\left(\frac{-e^{-x_0(\theta_1+\lambda)}}{(\theta_1+\lambda)(\theta_1-\theta_2+\lambda)^2}\right)+\left(\frac{.e^{-x_0\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)^2}\right)\\-\left(\frac{.x_0e^{-x_0\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)}\right)-\left(\frac{.e^{-x_0\theta_2}}{\theta_2^2.(\theta_1-\theta_2+\lambda)}\right)\end{array}\right\} \quad (13)$$

$\frac{d}{ds}\left(I_1+I_2+I_3\right)$ =0 gives,

$$= \frac{h\theta_1}{\theta_1+\lambda}\left\{\left[1-e^{-S(\theta_1+\lambda)}\right]\right\} + d\theta_1\left\{\left[\frac{.e^{-x_0(\theta_1+\lambda)}}{\theta_1+\lambda}\right] - \left[\frac{.e^{-S(\theta_1+\lambda)}}{(\theta_1+\lambda)}\right]\right\}$$

$$+d\lambda\theta_2^2\left\{\begin{array}{l}\left(\dfrac{-e^{-x_0(\theta_1+\lambda)}}{(\theta_1+\lambda)(\theta_1-\theta_2+\lambda)^2}\right)+\left(\dfrac{.e^{-x_0\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)^2}\right)\\ -\left(\dfrac{.x_0e^{-x_0\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)}\right)-\dfrac{.e^{-x_0\theta_2}}{\theta_2^2.(\theta_1-\theta_2+\lambda)}\end{array}\right\}=0$$

$$e^{-S(\theta_1+\lambda)}\frac{\theta_1}{\theta_1+\lambda}[h+d] = \frac{\theta_1}{\theta_1+\lambda}\left[h+d.e^{-S(\theta_1+\lambda)}\right]+$$

$$d\lambda\theta_2^2\left\{\begin{array}{l}\left(\dfrac{-e^{-x_0(\theta_1+\lambda)}}{(\theta_1+\lambda)(\theta_1-\theta_2+\lambda)^2}\right)+\left(\dfrac{.e^{-x_0\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)^2}\right)\\ -\left(\dfrac{.x_0e^{-x_0\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)}\right)-\dfrac{.e^{-x_0\theta_2}}{\theta_2^2.(\theta_1-\theta_2+\lambda)}\end{array}\right\} \quad (14)$$

Any value of S, which satisfies (14), is the optimal S.

Numerical Illustrations:

Case i)

$\theta_1 = 1.5$: $\theta_2 = 2.5$: $\lambda = 1$: $x_0 = 20$: $h = 5$

d	10	20	30	40	50
S	0.4394	0.6438	0.7784	0.8789	0.9592

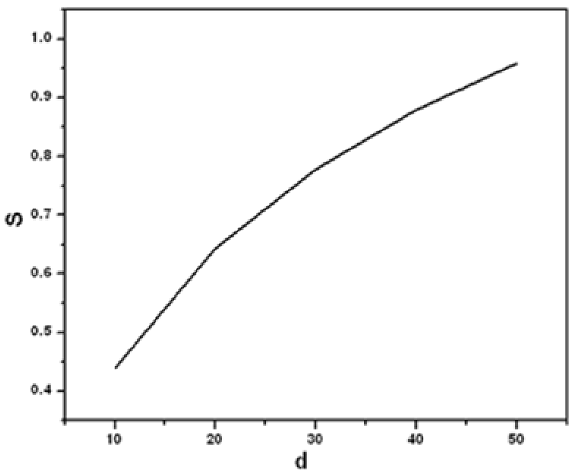

Fig. 5:

Case ii)

$\theta_1 = 1.5$: $\theta_2 = 2.5$: $\lambda = 1$: $x_0 = 20$: $d = 10$

h	5	10	15	20	25	30
S	0.4394	0.2773	0.2043	0.1622	0.1346	0.1151

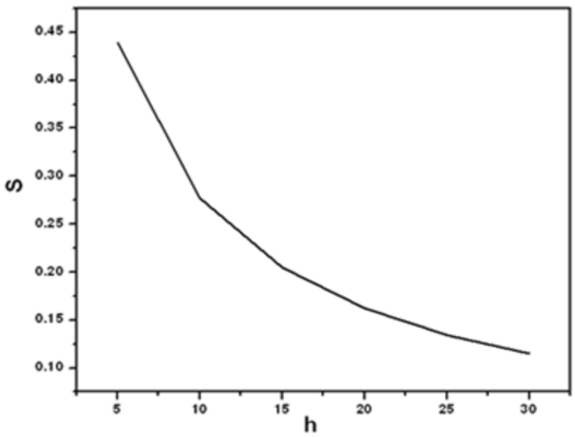

Fig. 6:

Inferences:

In *Case i*, it is seen that as the value of *d* increases the shortages should be avoided. Hence the size of inventory is made larger. This is very well suggested in this model also.

In *Case ii*, it may be noted that as the inventory holding cost 'h' increases a marginal decrease in the size $\hat{S}$ is suggested.

Model 2b): $x_0 > S$

$$E(C) = h\int_0^{x_0}(S-X)f_1(x)dx + h\int_{x_0}^{S}(S-X)f_2(x)dx + d\int_S^{\infty}(X-S)f_2(x)dx \tag{15}$$

$\frac{dE(c)}{dS} = 0$ gives,

$$-d\lambda\theta_2^2\left\{\begin{matrix}\left[\frac{e^{-s(\theta_1+\lambda)}}{(\theta_1+\lambda)(\theta_1-\theta_2+\lambda)^2}\right]-\left[\frac{e^{-s\theta_2}}{\theta_2(\theta_1-\theta_2+\lambda)^2}\right] \\ +\left[\left(\frac{s.e^{-s\theta_2}}{\theta_2.(\theta_1-\theta_2+\lambda)}\right)+\left(\frac{.e^{-s\theta_2}}{\theta_2^2.(\theta_1-\theta_2+\lambda)}\right)\right]\end{matrix}\right\} = 0 \tag{16}$$

Any value of S, which satisfies (16), is the optimal S.

Numerical Illustrations:

Case I)

$\theta_1 = 1$: $\theta_2 = 2.5$: $\lambda = 1$: $x_0 = 20$: $d = 50$

H	5	7	9	11	13
S	5.2819	3.9752	3.055	1.8613	0.7645

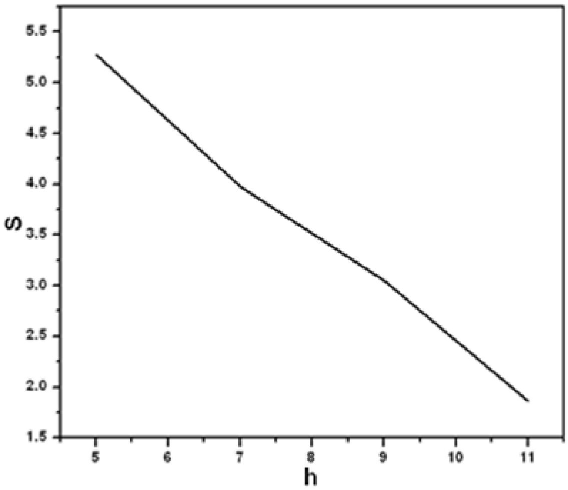

Fig. 7:

Case ii)

$\theta_1 = 1.5$: $\theta_2 = 2.5$: $\lambda = 1$: $x_0 = 20$: $h = 5$

D	10	14	18	22
S	5.2819	7.3757	9.572	10.4859

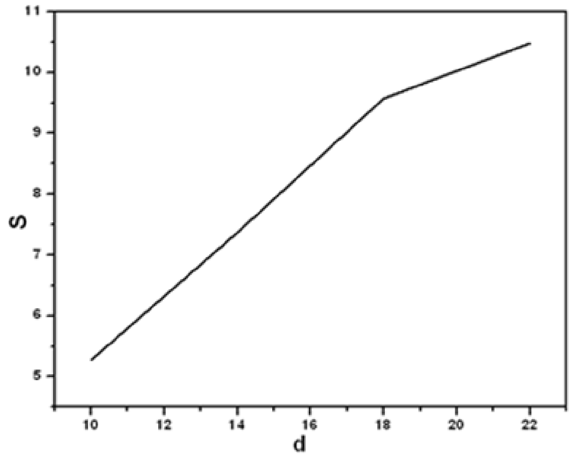

Fig. 8:

***Inferences*:**

In *Case i*, it is seen that as the inventory holding cost to *h* increases then the model suggested a smaller inventory size which is very common to all inventory models.

In *Case ii*, it is seen that as the value of shortage cost'd' increases, in this model also larger inventory size is suggested as in the case of all the models.

Conclusion:

In all the studies of our models, the inventory holding cost 'h' increases with decrease in S, the supply size or initial stock level for a smaller inventory size and the inventory shortage cost'd' increases with S for larger inventory size. Hence from this study, it is clear that the truncation point model is a very powerful methodology to study the difference in behavior such as before truncation point it is exponential distribution and after truncation point it is Erlang2 distribution.

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