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Hypothesis Tests For Marshall-Olkin Bivariate Exponential Distribution By Optimizing Posterior Mode

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Abstract: In this paper independency and symmetric tests for Marshall-Olkin (M-O) bivariate exponential model (BVEM) were interested. Tests have been conducted through optimizing the posterior modes of parameters of M-O BVEM. Simulation study has given as evidence to support our computations.

Key words: Bivariate exponential, Marshall-Olkin, Optimizing Posterior, Simulation study

INTRUDUCTION

The Bivariate survival data arise when two components with inter-corrolated are interested in study. Failure times of paired human organs, e.g. , kidneys, eyes, lungs and the recurrences of a given deceases (acute and chronic disease) are particular examples of such data. In industrial application these data types may come from systems whose survival depends on the survival of two similar components. Marshall-Olkin (M-O) (1967) proposed bivariate exponential distribution (BVED) for failure time distribution of paired components on which two components can fail simultaneously. This model satisfy the loss of memory property (LMP) and the marginals are exponential distribution but it is not absolutely continuous with respect to lebesgue measure in R_2 . The p.d.f. of BVED is given by

$$f(x,y)=\begin{cases}\lambda_1(\lambda_2+\lambda_3)\exp(-\lambda_1 x-(\lambda_2+\lambda_3)y), & 0<x<y\\ \lambda_2(\lambda_1+\lambda_3)\exp(-\lambda_2 y-(\lambda_1+\lambda_3)x), & 0<y<x\\ \lambda_3\exp(-\lambda x), & x=y\end{cases}$$

Where $\lambda=\lambda_1+\lambda_2+\lambda_3$.

Bemis, Bain and Higgins (1972) suggested test for independent for BVED, Bhattacharya and Johnson (1973) showed that the test suggested by Bemis, et al., (1972) is in fact the UMPU test, Awad, Azzam and Hamdan (1981) suggested test for symmetry through likelihood ratio test.

Gupta, Mehrotra and Michalek (1984) suggested generalized likelihood ratio test (GLRT) for independence in BVED. Also Hanagal(1992), Hanagal and Kale(1991a,1991b,1992)have done extensive tests for testing symmetry and independence for four BVED. Recently Ahmadi and Hanagal (2008) developted Bayesian test base on B-factor for symetric and independency tests purpose. In this paper, we suggest a new Bayesian method test through optimizing the posterior mode of M-O BVED models. The posterior mode is easy to obtain with adventage of Newton-Rafson method.

2-Bayes Inference and Asymptotic Methods:

Let (X,Y) indicates the bivariate random variables corresponding to the observed data, so (x,y) having BVED and its pdf is $f((x,y),\underline{\lambda})$ where $\underline{\lambda}$ is the vector of unknown parameters with parameter space Ω.
The kernel posterior pdf of parameters is well known as updating the parameters distribution of prior with advantage of data at hand presented as the liklihood part. Therefore the posterior pdf with canceling the free-parameters part which does not depend on the parameters has given as follows.

$$p(\underline{\lambda}|(X,Y))\propto L((X,Y),\underline{\lambda})p(\underline{\lambda}) \tag{1}$$

Where we the kernel prior were opted

$$p(\lambda_i,\alpha_i,\beta_i)\propto \lambda_i^{\alpha_i-1}\exp(-\beta_i\lambda_i) \tag{2}$$

Which is $Gamma(\alpha_i,\beta_i)$, also $L((X,Y),\underline{\lambda_i})$ is likelihood function of corresponding pdf of (M-O), Furthermore, the complete form of posterior is

$$p(\underline{\lambda}|(X,Y))\propto L((X,Y),\underline{\lambda})\prod_{i=1}^{k}\lambda_i^{\alpha_i-1}\exp(-\beta_i\lambda_i) \tag{3}$$

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Where k is the number of parameters on the pdf and λ_i are independent because they express intensity of three independent chocks M-O (1967).

Bereger(1985), Carlin B. and Louis T.(1996), Gelman A. and Carlin J. and Stern H. and Rubin D.(2004) have suggested the normal approximations of the posterior distribution. In this case suppose posterior $p(\underline{\lambda}|(X,Y))$ has unimodal and roughly symmetric, therefore, it is convenient to approximate it by a normal distribution centered at the mode. A Taylor series expansion of $\log(\underline{\lambda}|(X,Y))$ centered at the posterior mode $\hat{\underline{\lambda}}$ gives

$$\log(\underline{\lambda}|(X,Y)) = \log\left(\hat{\underline{\lambda}}|(X,Y)\right) + \left(\underline{\lambda} - \hat{\underline{\lambda}}\right)\left[\frac{\partial \log(\underline{\lambda}|(X,Y))}{\partial \underline{\lambda}}\right]_{\underline{\lambda}=\hat{\underline{\lambda}}} + \frac{1}{2}\left(\underline{\lambda} - \hat{\underline{\lambda}}\right)^T \left[\frac{\partial^2 \log p(\underline{\lambda}(X,Y))}{\partial \underline{\lambda}^T \partial \underline{\lambda}}\right]_{\underline{\lambda}=\hat{\underline{\lambda}}} \left(\underline{\lambda} - \hat{\underline{\lambda}}\right) \tag{4}$$

Where the second term (linear term) of above expression is being zero as the log posterior density attains its the maximum value as mode. The higher order term in (4) become negligible in importance relative to the quadratic term as $\underline{\lambda}$ is close to $\hat{\underline{\lambda}}$ and n is large.

In above expression (4) the first term is constant and the second term is proportional to the logarithm of normal density therefore $p(\underline{\lambda}|(X,Y)) \approx N\left(\hat{\underline{\lambda}}, \left[I\left(\hat{\underline{\lambda}}\right)\right]^{-1}\right)$ where

$$I\left(\hat{\underline{\lambda}}\right) = -\left.\frac{\partial^2 \log(\underline{\lambda}|(X,Y))}{\partial \underline{\lambda}^T \partial \underline{\lambda}}\right|_{\underline{\lambda}=\hat{\underline{\lambda}}} \tag{5}$$

3-Estimation Of the Hyper Parameters Via Consistent Estimators of BVED:

Suppose (X,Y) have joint density $f((x,y),\lambda)$ where λ is a vector of unknown parameters. Denote the prior distribution for λ by $p(\lambda,\psi)$ where ψ is the parameter of distribution of λ. First we obtain consistent estimators of each parameters of M-O BVED separately through moment method explained in next section, then their corresponding mean and variance were obtained though simulation sampling. Consequently the estimated value of α_i and β_i were obtained by solving two simple equation $E(\lambda_i) = \alpha_i / \beta_i$ and $V(\lambda_i) = \alpha_i / \beta_i^2$.

Where $E(\lambda_i)$ and $V(\lambda_i)$ are the same simulation mean and variance respectively. The similar way has been already presented by Hanagal and Ahmadi (2009).

One can perform bootstrap sampling instead of simulation when a real data set are available.

3.1 Marshall-Olkin Model:

The marginal distribution are obtained from the p.d.f.(1.1)

$$f(X) = (\lambda_1 + \lambda_3)\exp(-(\lambda_1 + \lambda_3)x), \quad x > 0$$
$$f(Y) = (\lambda_2 + \lambda_3)\exp(-(\lambda_2 + \lambda_3)y), \quad y > 0$$

The moment can easily obtain by $E(X) = \frac{1}{\lambda_1 + \lambda_3}, E(Y) = \frac{1}{\lambda_2 + \lambda_3}$ and $E(\min(X,Y)) = \frac{1}{\lambda}$ therefore the consistent estimators are given

$$\frac{n}{\sum_{i=1}^{n}\min(x_i,y_i)}-\frac{n}{\sum_{i=1}^{n}y_i}\xrightarrow{p}\lambda_1$$

$$\frac{n}{\sum_{i=1}^{n}\min(x_i,y_i)}-\frac{n}{\sum_{i=1}^{n}x_i}\xrightarrow{p}\lambda_2$$

$$\frac{n}{\sum_{i=1}^{n}x_i}+\frac{n}{\sum_{i=1}^{n}y_i}-\frac{n}{\sum_{i=1}^{n}\min(x_i,y_i)}\xrightarrow{p}\lambda_3$$

4-Marshal-Olkin (M-O) Model:

In (M-O) model the probability of simultaneous failure of both the components is $P(X=Y)=\frac{\lambda_3}{\lambda}$. It also can consider as the correlation coefficient between X,Y. Furthermore, if $\lambda_3=0$ it indicates that X,Y are independent and $\lambda_1=\lambda_2$ also determine that the p.d.f. has symmetric character. Therefore, two tests are prominent.

1-Test for Independence: $H_0:\lambda_3=0$ vs $H_1:\lambda_3>0$:

Based on section 2 it can be yielded that $\hat{\lambda}_3$ has asymptotically normal, $AN(\lambda_3, I^{33}/n)$ therefore an approximation test for independency under H_0 is obtained by $T_1=\sqrt{n}\,\hat{\lambda}_3\Big/\sqrt{\hat{I}^{33}}$ where $\hat{I}^{33}$ is obtained by substituting the posterior modes of parameters in I^{33}.

We reject H_0 if $T_1>z_{1-\alpha}$.

2-Test for Symmetry:

$H_0:\lambda_1=\lambda_2=\lambda_0$ (Known) vs. $H_1:\lambda_1\neq\lambda_2$:

Again from the result of section 2 we have $\hat{\lambda}_2-\hat{\lambda}_1$ has asymptotically normal, $AN(\lambda_2-\lambda_1,\sigma^2(I)/n)$ where $\sigma^2(I)=I^{22}+I^{11}-2I^{12}$. Test of symmetric under H_0 is $T_2=\sqrt{n}\left(\hat{\lambda}_2-\hat{\lambda}_1\right)\Big/\hat{\sigma}(I)$ and T_2 has standard normal distribution. H_0 is rejected if $T_2>z_{1-\alpha/2}$.

5 Numerical Simulation Study:

In each model of BVED, 1000 batches of observations of size 50, 75 and 100 were generated for Marshall-Olkin model; the parameters of prior distribution were estimated based on consistent estimators section 3. The generated data were used to obtain posterior modes then sections 4, 5 were used for testing hypothesis purpose. Power of test also has been obtained in different level of significance. We observed that the power of test was increased as sample size was increased. We present our results in the following Tables 7.1-7.2

Table 7.1: Test for independence of parameters of BE of M-O.

M-O						
Parameter	λ_1	λ_2	λ_3	$\hat{\alpha}_3$	$\hat{\beta}_3$	Power
n=40	0.08	0.1	0.03	1.152	37.21	0.865
n=50	0.08	0.1	0.03	1.315	42.75	0.942
n=60	0.08	0.1	0.03	1.45	47.5	0.945

Table 7.2: Test for Symmetry of Parameters of BVE of M-O

M-O								
Parameters	λ_1	λ_2	λ_3	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	Power
n=40	0.05	0.1	0.03	1.84	36.11	2.6	16.87	0.969
n=50	0.05	0.1	0.03	2.01	39.43	2.7	17.48	0.992
n=60	0.05	0.1	0.03	2.13	41.92	2.72	17.845	0.996

Discussions:

We tried to develop Bayesian hypothesis test via maximization of posterior mode for M-O, BVED and showed that the implementation of Bayesian method for independence and symmetry tests were easy to compute. Choosing the Gamma priors had advantage on computations and was tractable. Furthermore one can verify that the nature of M-O pdf has exponential distribution so the computation expressions were suitable as Gamma prior was selected. We also proposed to use consistent estimators to estimate the hyper-parameters which in turn had an extension of Empirical Bayes method. One can apply this method by using past data to estimate highper parameters as prior and then try to conduct the independency and symmetric tests which in turn leading to performing test with impact of past data.

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