On Geophysical Application of the Separation Between the Geoid and the Quasi-geoid Case Study: Four Regions in Iran

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Abstract: In this paper the separation between the geoid and the quasi-geoid over 4 test regions, namely, Alborz and Zagros mountain ranges (as the two test areas over mountainous regions) and the coast of Caspian Sea and Khuzestan province (as the two test areas over flat regions) is computed. The result of the computations shows that average and standard deviation of the separation between the geoid and the quasi-geoid are, respectively, -488.586 (mm) and 265.031 (mm) over Alborz mountain, -731.654 (mm) and 322.520 (mm) over Zagros mountain, -0.760 (mm) and 5.130 (mm) over the coast of Caspian Sea, and -5.076 (mm) and 10.857 (mm) over the Khuzestan province. Furthermore, by using the EGM2008 geopotential model the geoidal undulations over the measured points of the aforementioned test areas are computed and by knowing the separation between the geoid and the quasi-geoid, the quasi-geoid over the test areas is derived. Based on the numerical results the separation between the geoid and the quasi-geoid in addition of its application for the transformation of normal heights to orthometric heights, and vice versa, it can be used as a tool for the study of the variation of crustal density and prevailing isostasy mechanisms from one region to another.

Key words: Separation between the geoid and the quasi-geoid, geoid, quasi-geoid, normal height, orthometric height.

INTRODUCTION

The main goal of this paper is to show that the separation between the geoid and the quasi-geoid besides its traditional application for the transformation of orthometric heights into the normal heights and vice-versa can be used to extract information about the variation of crustal density and prevailing isostasy mechanisms from one region to another. Geoid as defined by Gauss (1828) and Listing (1873) is an equipotential surface of the Earth’s gravity field that fits to Mean Sea Level (MSL) in an optimum way (i.e. least squares sense), and has been selected by the geodetic community as the reference surface for the orthometric height system. Molodensky (1945, 1948, 1960) defined the telluroid as a mathematical surface on which the normal potential of points is equal to the actual potential of corresponding points on the surface of the earth. More precisely is shown in Fig.1 the actual potential at point $P$ on the surface of the Earth is equal to the normal potential at point $P'$ on the surface of the telluroid ($W_P=U_P$). For telluroid computation we must also define a projection scheme between the corresponding points on the surface of the earth and the telluroid. Molodensky used projection along the normal to the reference ellipsoid. The separation between the Earth’s surface and the telluroid is known as “telluroidal height” or “height anomaly”($\zeta$), see Fig. 1. If this separation be placed on the surface of a reference ellipsoid we arrive at a surface known as the “quasi-geoid”. The quasi-geoid is the reference surface of the normal height system. Nowadays, the height system of the countries around the world is either orthometric or normal height. Therefore it might be needed to transfer the heights from one system to another.

In the next section (Section 2) we will go into details of the transformation between these two height systems. Next, in Section 3, the presented formulas will be applied to four test areas in mountainous and flat regions within the territory of Iran. Finally, Section 4 will provide the conclusions.
2. Computation of the Separation Between the Geoid and the Quasi-geoid:
To begin with the computation of the separation between the geoid and the quasi-geoid let us once again refer to Fig.1. Accordingly we can write:

\[ H^O + N = H^N + \zeta \]  \hspace{1cm} (1)

or

\[ N - \zeta = H^N - H^O \]  \hspace{1cm} (2)

The orthometric height \( H^O \) and the normal height \( H^N \) are defined as follows:

\[ H^O = \frac{C}{\bar{g}} \]  \hspace{1cm} (3)

\[ H^N = \frac{C}{\bar{\gamma}} \]  \hspace{1cm} (4)

Where \( C = W_o - W \) is the geopotential number, i.e., the geoid’s potential value \( W_o \) minus the potential value \( W \) at a computational point \( P \). \( \bar{g} \) is the mean actual gravity along the plumb line from the surface of the earth down to the geoid. \( \bar{\gamma} \) is the mean normal gravity along the normal from the telluroid to the surface of the reference ellipsoid, or equivalently from the surface of the earth down to the quasi-geoid. Upon the substitution for \( H^O \) and \( H^N \) in Eq. (2) from Eqs. (3) and (4), respectively, we arrive at:

\[ N - \zeta = \frac{C}{\bar{g}} \cdot \frac{C}{\bar{g}} \]
\[ = C\left(\frac{1}{\bar{g}} - \frac{1}{\bar{g}}\right) \]
\[ = C\left(\frac{\bar{g} - \bar{\gamma}}{\bar{g} \cdot \bar{\gamma}}\right) \]  \hspace{1cm} (5)

Now if we substitute for \( C \) in Eq. (5) from Eq. (3) we have:

\[ N - \zeta = H^O \cdot \frac{\bar{g} - \bar{\gamma}}{\bar{g} \cdot \bar{\gamma}} \]
\[ = H^O \left(\frac{\bar{g} - \bar{\gamma}}{\bar{g}}\right) \]  \hspace{1cm} (6)
The relation shown in Eq. (6) in some literatures is referred to as “C2-Term” (Featherstone and Kirby 1998, Nahavandchi 2002). In the case of the normal height computation, the normal field is generated by the Somigliana-Pizzetti reference field (Pizzetti 1894, Somigliana 1930). The term \( g \gamma \) is mainly due to the masses above the geoid, known as the “topographic masses”. If we approximate the effect of topographical masses by the Bouguer anomaly, \( \gamma \) can be approximated by \( \Delta g_B \) (see Appendix for the proof) and Eq. (6) can be written as follows:

\[
N - \zeta = \left( \frac{\Delta g_B}{\gamma} \right) H^O
\]  

(7)

An approximate formula for the computation of \( \gamma \) according to Hofmann-Wellenhof and Moritz (2006) is as follows:

\[
\gamma = \gamma \left[ 1 - \left( 1 + f + m - 2 f \sin^2 \phi \right) \frac{H^O}{a} + \frac{H^O}{a^2}\right]
\]  

(8)

Where \( \gamma \) is the normal gravity of the Somigliana-Pizzetti type on the surface of the reference ellipsoid, \( f \) is the geometric flattening, \( m \) is the Clairaut’s constant, \( a \) is the semi major axis of the reference ellipsoid and \( \phi \) is the geodetic latitude of the computational point.

Having computed \( N - \zeta \) (the “C2-Term”) by knowing the orthometric height \( H^O \), the normal height \( H^N \) can be derived from Eq. (1) as follows:

\[
H^N = H^O + (N - \zeta)
\]  

(9)

Or again by using Eq. (1) knowing the “C2-Term” and the geoidal undulation \( N, \zeta \) can be derived as follows:

\[
\zeta = N + (H^O - H^N)
\]  

(10)

Having furnished the definition and formulas related to the separation between the geoid and the quasi-geoid we will proceed to the next section where we will present four test computations and will discuss on the numerical results.

3. Test Computations and Analysis of Results:

In this section, we will present the results of our test computation for derivation of the separation between the geoid and the quasi-geoid within four test regions. Besides, using the equations (9) and (10) the normal heights and the height anomalies, respectively, are computed for the four test regions. These test regions are located at Alborz mountain range \((48.667^\circ \leq \lambda \leq 55.333^\circ, 35.167^\circ \leq \phi \leq 37^\circ)\) and Zagros mountain range \((48.5^\circ \leq \lambda \leq 52^\circ, 30^\circ \leq \phi \leq 33.5^\circ)\) as highland areas with the heights more than 800 m and two areas at the coast of Caspian Sea \((50.333^\circ \leq \lambda \leq 55^\circ, 36.5^\circ \leq \phi \leq 37^\circ)\) and the Khuzestan province \((48^\circ \leq \lambda \leq 50^\circ, 30^\circ \leq \phi \leq 32^\circ)\) as lowland areas with the heights less than 400 m. The data used for the four test computations are comprised of gravity and orthometric height over 1471 points in Alborz, 1224 points in Zagros, 267 points in Caspian Sea coast and 552 points in Khuzestan province. Table 1, provides some statistical information about the height variations in the test areas. These data are provided by the kind grace of Iranian National Cartographic Center (NCC).

<table>
<thead>
<tr>
<th>Test regions</th>
<th>Max</th>
<th>Min</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alborz mountain</td>
<td>3743.43</td>
<td>800.93</td>
<td>1680.717</td>
</tr>
<tr>
<td>Zagros mountain</td>
<td>3585.29</td>
<td>800.22</td>
<td>2014.63</td>
</tr>
<tr>
<td>Caspian Sea coast</td>
<td>384.32</td>
<td>-29</td>
<td>10.699</td>
</tr>
<tr>
<td>Khuzestan Province</td>
<td>389.02</td>
<td>-0.01</td>
<td>41.084</td>
</tr>
</tbody>
</table>
For the computation of the separation between the geoid and the quasi-geoid according to formula (7) in the mountainous test regions due to the rough topography, the complete Bouguer anomaly, i.e., the Bouguer plate plus the topographic correction, is used while for the Caspian Sea coast and Khuzestan province test regions the simple Bouguer anomaly, i.e. Bouguer plate, is considered. It is due to the facts that contribution of the topographic corrections to the separation between the geoid and the quasi-geoid is estimated to be 2-3 cm in very rugged areas (Flury and Rummel 2009). We also used the constant value $2.67 \text{ (gr/cm}^3\text{)}$ for the density of the crust in the computations of the Bouguer anomaly ($A_{B}$).

The separation between the geoid and the quasi-geoid as well as height anomalies and normal heights derived for the test areas are shown in Fig. 2 – Fig. 7 for the Alborz Mountains and Khuzestan Province as samples of our results for highland and lowland areas, respectively.

The summary statistical information of the computed separation between the geoid and quasi-geoid within the four test areas are given in Table 2.
Fig. 5: The separation between the geoid and the quasi-geoid within Khuzestan Province test area in millimeter (contour intervals 2 mm)

Fig. 6: Height anomalies within Khuzestan Province test area in meters (contour intervals 0.25 m)

Table 2: Summary of statistical information of the separation between the geoid and the quasi-geoid within the four test regions

<table>
<thead>
<tr>
<th>Test Area</th>
<th>Max. (mm)</th>
<th>Min. (mm)</th>
<th>Mean (mm)</th>
<th>STD (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alborz Mountains (Highland)</td>
<td>28.746</td>
<td>-1460.818</td>
<td>-488.586</td>
<td>265.031</td>
</tr>
<tr>
<td>Zagros Mountains (Highland)</td>
<td>-120.077</td>
<td>-1527.612</td>
<td>-731.654</td>
<td>322.52</td>
</tr>
<tr>
<td>Caspian Sea Coast (Lowland)</td>
<td>9.834</td>
<td>-41.292</td>
<td>-0.76</td>
<td>5.13</td>
</tr>
<tr>
<td>Khuzestan Province (Lowland)</td>
<td>0.0006</td>
<td>-65.498</td>
<td>-5.076</td>
<td>10.857</td>
</tr>
</tbody>
</table>
According to Table 2, the separation between the geoid and the quasi-geoid is most pronounced in mountainous areas (compare the maximum separation of 1.461 m and 1.528 m in Alborz and in Zagros Mountains, respectively, with the maximum values of 41.292mm and 65.498mm in Caspian Sea coast and Khuzestan province, respectively). The negative values of the separation between the geoid and the quasi-geoid within the Zagros Mountain test area indicate that the quasi-geoid is higher than the geoid in that region. Besides, the computed separations within the Alborz Mountain, Caspian Sea coast and Khuzestan province are 99.8%, 27.3% and 95.7% negative, respectively.

Furthermore, in Fig. 8 and Fig. 9 we have examined the correlation of the computed separation of the geoid and the quasi-geoid with the heights within the four test regions.

**Fig. 7:** Normal heights within Khuzestan Province test area in meters (contour intervals 30 m)

**Fig. 8:** The Pearson Correlation Coefficient of the separation between “the geoid and the quasi-geoid” and the heights within (a) Alborz Mountains and (b) Zagros Mountains as our highland test areas
A glance to Fig. 8 and Fig. 9 shows that the separation between the geoid and the quasi-geoid is reduced by increasing the topographic heights. Indeed, the absolute value of the separation between the geoid and the quasi-geoid is increased by increasing the heights. Besides, from Fig. 8 and Fig. 9 we can clearly see that the rate of change of the separation between the geoid and the quasi-geoid in the highland test regions is much greater than the lowlands. In other words the geoid and quasi-geoid in lowland regions are closer to each other as compared to the highland test regions. The same conclusion can be drawn from Fig. 2 - Fig. 4 corresponding to the highland test region and Fig. 5 – Fig 7 which are corresponding to lowland region. As a matter of fact, in the places with higher slope rates, we see sharper variations in the changes of the height anomalies as well as the separations between the geoid and the quasi-geoid.

On the other hand, the comparison of the four graphs shown in Fig. 8 and Fig 9 suggests that even though the separation between the geoid and the quasi-geoid is highly correlated with the heights, however, this correlation can vary from one region to another (compare the correlation coefficient of 0.6788 associated to the Caspian Sea coast, one of our lowland test regions, with the value of 0.9775, which corresponds to Khuzestan province as the other test region at the lowland region in our test computations). It is important to note that the heights within these two regions are below 400 m and the two regions are having almost the same topography. This can also be seen in the results of the highland test regions as well. To explain the difference among the computed correlation coefficients we must once again refer to Eq. (7). According to this equation the separation between the geoid and the quasi-geoid is related not only to the orthometric heights but also to the Bouguer anomalies, which in turns are also affected by the applied hypothetical value for the crustal density. Therefore, it can be concluded that the computation of the separation between the geoid and the quasi-geoid not only provides a tool for the transformation from normal heights to orthometric heights and vice versa, but also can be used as a means for getting information about the crustal density variation from one region to another. Besides, the difference among the calculated correlation coefficients can be partially due to the difference in the prevailing isostasy mechanisms. For example, here the distinct difference between the correlation coefficients of the separation between the geoid and the quasi-geoid with orthometric heights can be associated with the different geological structure of the two regions, which has already been reported by Dehghani and Makris (1983), Shahabpour (1999) and Mokhtari et al. (2004).
Conclusions:

In this paper we offered the detailed computational procedure for calculation of the separation between the geoid and the quasi-geoid. Besides, we showed that this separation is highly correlated with topography. However, since the separation between these two surfaces is due to the masses between the geoid and the surface of the Earth, the computed separation is influenced by mixed effect of the topographic heights, mass density variations within the topographic layers, and the isostasy mechanism. Therefore, the computation of the separation between the geoid and the quasi-geoid besides its application for the transformation of the normal heights to the orthometric heights and vice versa, it also can be used for obtaining information about the variation of the geophysical and the geological structures of the crust from one region to another.

REFERENCES


Appendix: Approximation of the effect of topographic masses (g - g) by (Δg). (The proof)

The observed gravity value gobs on the surface of the Earth can be reduced to the gravity value gGeoid at the geoid’s level as follows:

\[ g_{\text{Geoid}} = g_{\text{obs}} + \delta g_f - 2\delta g_B \]

Where \( \delta g_f \) and \( \delta g_B \) are free-air and Bouguer reductions, respectively. On the other hand according to Prey-Poincare reduction, the mean gravity value \( \overline{g} \) from the surface of the Earth down to the geoid’s level can be derived as follows:

\[ \overline{g} = \frac{g_{\text{obs}} + g_{\text{Geoid}}}{2} = \frac{g_{\text{obs}} + g_{\text{obs}} + \delta g_f - 2\delta g_B}{2} \]
Here the variations of the gravity value from the surface of the Earth down to the geoid’s level is considered linear. Substituting the $\delta g_b$ in the above formula by $1/3 \delta g_f$, we have:

$$\bar{g} = g_{obs} + \frac{1}{2} \delta g_f - \frac{1}{3} \delta g_f$$

Now the normal gravity on the surface of the telluroid $\gamma_{tell}$, knowing the normal height $H_N$ and the normal gravity value on the surface of the reference ellipsoid $\gamma_0$, can be computed from the linear part of the Taylor expansion of the normal gravity on the surface of the telluroid, as follows:

$$\gamma_{tell} \approx \gamma_0 - \frac{\partial \gamma}{\partial h} H_N \approx \gamma_0 - 0.3086 H^O \approx \gamma_0 - \delta g_f$$

In the above formula we have also approximated the normal height $H_N$ by the orthometric height $H^O$.

Finally, the mean normal gravity from the surface of the telluroid to the surface of the reference ellipsoid, by assuming the linearity of the vertical variations of the normal gravity, can be computed as follows:

$$\bar{\gamma} = \frac{\gamma_0 + \gamma_{tell}}{2} = \frac{\gamma_0 + \gamma_0 - \delta g_f}{2} = \gamma_0 - \frac{1}{2} \delta g_f$$

Therefore, \(\bar{g} - \bar{\gamma}\) can be derived as follows:

$$\bar{g} - \bar{\gamma} = g_{obs} + \frac{1}{2} \delta g_f - \frac{1}{3} \delta g_f - \gamma_0 + \frac{1}{2} \delta g_f$$

$$= g_{obs} + \delta g_f - \frac{1}{3} \delta g_f - \gamma_0$$

$$= g_{obs} + \delta g_f - \delta g_B - \gamma_0$$

$$= \Delta g_B$$

and finally we obtain:

$$\bar{g} - \bar{\gamma} = \Delta g_B$$