Computing of Some Topological Indices of Corona Product Graphs

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Abstract: Let $G = (V, E)$ be a simple graph. In this paper, we supply exact formulas for the Wiener, PI, vertex PI, edge Szeged and edge-vertex Szeged indices of the Corona product graphs. In addition, we give some example for these indices over the Corona product graphs.

Key words: Wiener index, Edge Szeged index, PI index, Corona product.

INTRODUCTION

A graph $G$ is defined as a pair $G = (V, E)$, where $V$ is a non-empty set of vertices and $E$ is a set of edges. Throughout this paper graphs are simple and connected. We will deal with finite graph, i.e., both $|V|$ and $|E|$ are finite sets. A topological index is a real number related to a graph, it does not depend on the labeling or the pictorial representation of a graph. There are several topological indices. Among them are the Wiener index $W$, the Randić index $R$, the Hosoya index $Z$, the Merrifield-Simmons index $W$, the Szeged index $Sz$ and the vertex and edge Padmakar-Ivan indices $PI_v$ and PI. These topological indices have found applications as means for modeling chemical, pharmaceutical and other properties of molecules. For more results on topological indices of graphs see, (Klavžar et al., 1996), (Gutman et al., 1997), (Cash, 2002) and (Eliasi, 2009).

As usual, we denote the distance between two arbitrary vertices $u$ and $v$ of $G$ by $d_G(u, v)$ (short). It is defined as the number of edges in the minimal path connecting the vertices $u$ and $v$. The degree of $u$, denoted by $d_G(u)$, is the number of edges incident with $v$ in $G$. The Wiener index of graph $G$ is defined as $W(G) = \sum_{(u, v) \in E(G)} d_G(u, v)$. We assume that $e = uv$ is an edge connecting the vertices $u$ and $v$. Suppose that $n_u(e, G)$ is the number of vertices of $G$ lying closer to vertex $u$ than to vertex $v$ and that $n_v(e, G)$ is the number of vertices of $G$ lying closer to vertex $v$ than to vertex $u$. Vertices equidistant to $u$ and $v$ are not taken into account. The Szeged index of graph $G$ is defined as $Sz(G) = \sum_{e=uv \in E} n_u(e, G)n_v(e, G)$ and the vertex PI index of graph $G$ is defined as:

$$PI_v(G) = \sum_{e=uv \in E} [n_u(e, G) + n_v(e, G)]$$

If $e = xy$ is an edge of $G$, then we defined $d(u, e) = \text{Min}\{d(u, x), d(u, y)\}$. An edge version of the Szeged index was introduced (Gutman et al., 2002) named "edge Szeged index". This index is defined as $Sz_e(G) = \sum_{e=uv \in E} m_u(e, G)m_v(e, G)$, where $m_u(e, G)$ is the number of edges lying closer to $u$ than $v$ and where $m_v(e, G)$ is defined analogously. The PI index of graph $G$ is defined as:

$$PI(G) = \sum_{e=uv \in E} [m_u(e, G) + m_v(e, G)]$$

We also introduce a further edge-vertex Szeged denoted here by $Sz_{ev}(G)$. It is defined as:

$$Sz_e(G) = \frac{1}{2} \sum_{e=uv \in E} [n_u(e, G)m_v(e, G) + n_v(e, G)m_u(e, G)]$$

The Wiener indices of the Cartesian products of graphs are determined in (Graovac et al., 1991) and the hyper-Wiener indices of the Cartesian products and the composition of graphs are determined in (Khalifeh et al., 2008). Moreover, in (Khalifeh et al., 2008) the vertex and edge PI indices of Cartesian product graphs and edge
Szeged index of product graphs are studied and the $GA_2$ index of some graph operations are determined in (Fath-Tabar at al., 2010). In this paper, we supply exact formulas for the Wiener, PI, vertex PI, edge Szeged and edge-vertex Szeged indices of the Corona product graphs and we give some example for these indices over the Corona product graphs.

Let $G$ and $H$ be two graph with sets vertex $V(G)$ and $V(H)$ and sets edge $E(G)$ and $E(H)$ respectively. For graphs $G$ and $H$ we defined the Corona product $G \circ H$ of two graphs $G$ and $H$ is obtained by taking $|V(G)|$ copies of $H$ and joining each vertex of $i$-th copy with vertex $v_i \in V(G)$, see (Fath-Tabar at al., 2010).

**Result 1:**
Obviously for the Corona product $G$ and $H$, we have $|V(G \circ H)| = |V(G)|(|V(H)| + 1)$ and $|E(G \circ H)| = |E(G)| + |V(G)||V(H)| + |E(H)|$.

**Result 2:**
For set of edges $G \circ H$, we have $E_1 = \{e \in E(G \circ H) | e \in E(G)\}$, $E_2 = \{e \in E(G \circ H) | e \in E(H_i), i = 1, 2, \ldots, |V(G)|\}$ and $E_3 = \{e = uv \in E(G \circ H) | u \in E(H_i), i = 1, 2, \ldots, |V(G)|, v \in V(G)\}$. It is easy to see that $E_1, E_2$ and $E_3$ are partition of the edge set of $G \circ H$ and also $|E_1| = |E(G)|$, $|E_2| = |E(H)||V(G)|$ and $|E_3| = |V(H)||V(G)|$.

Let $\{G_i\}_{i=1}^d$ be a set of finite pairwise disjoint graphs with $v_i \in V(G_i)$. The bridge graph $B(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d)$ of $\{G_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_d$ by connecting the vertices $v_i$ and $v_{i+1}$ by an edge for all $i = 1, 2, \ldots, d-1$, (Fig.1).

**Fig. 1:** The bridge graph.

We define $G_n(H, v) = B(H, H, \ldots, H; v, v, \ldots, v)$ (n times) which is the special case of bridge graph.

Clearly, $G_1(H, v) = H$ for any vertex $v$ of $H$. For example, let $P_m$ be the path graph on $m$ vertices $v_1, v_2, \ldots, v_m$, define $B_n = G_n(P_3, v_2)$, (see Fig. 2) (Polyethene when $n = 4$). As another example, let $C_k$ be the cycle with $k$ vertices and define $T_n = G_n(C_k, v_1)$, (see Fig. 3) when $k = 3$ and $n = 5$. As a final example, define the bridge graph $J_{n, m+1} = G_n(W_{m+1}, v_1)$, where $W_{m+1}$ be the Wheel graph on $m + 1$ vertices $v_1, v_2, \ldots, v_m, v_{m+1}$, such that $\deg(v_i) = m$ and $\deg(v_{i+1}) = 3$ for $i = 1, 2, \ldots, m + 1$.

**Fig. 2:** The graph $B_n$.

**Fig. 3:** The graph $T_{n, 3}$. 

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Result 3:
By definition, both Corona product graphs and bridge graph, implies that:
\[ B_n = P_n \circ K_2. \]
\[ T_{n,3} = P_n \circ K_2. \]
\[ J_{n,m+1} = P_n \circ C_m. \]

RESULTS AND DISCUSSION

The Wiener Index of the Corona Product Graphs:
In this section we compute the Wiener index of the corona product graph which is a main result of this paper.

Lemma 1.1:
Let \( G \) and \( H \) be two graphs. Then:
1. If \( u, v \in V(G) \), then \( d_{G \circ H}(u, v) = d_G(u, v) \).
2. If \( u \in V(G) \) and \( v \in V(H_i), i = 1, 2, \ldots, |V(G)| \), then \( d_{G \circ H}(u, v) = 1 + d_G(u, w_i) \), where \( w_i \) is the \( i \)-th vertex in \( G \).
3. If \( u, v \in V(H_i), i = 1, 2, \ldots, |V(G)| \), then \( d_{G \circ H}(u, v) = 1 \) whenever \( uv \in E(H) \) and \( d_{G \circ H}(u, v) = 2 \) whenever \( uv \notin E(H) \).
4. If \( u \in V(H_i), \ v \in V(H_j) \) and \( i \neq j \), then \( d_{G \circ H}(u, v) = 2 + d_G(w_i, w_j) \) where \( w_i \) and \( w_j \) are the \( i \)-th and \( j \)-th vertices of \( G \), respectively.

Theorem 1.2:
Let \( H \) be free-triangle and \( r \)-regular graph. Then the Wiener index of the corona product \( G \) and \( H \) is
\[ W(G \circ H) = (1+V(H))W(G)+|V(G)|V(H)+2|V(H)|+2\left|\frac{|V(G)|}{2}\right|+|V(G)||E(H)|. \]

Proof:
By the above lemma:
\[
W(G \circ H) = \sum_{u,v \in V(G \circ H)} d_{G \circ H}(u,v) = \sum_{u,v \in V(G)} \sum_{i=1}^{|V(G)|} d_{G \circ H}(u,v) + |V(G)| \sum_{i=1}^{|V(G)|} d_{G \circ H}(u,v) + |V(G)| \sum_{u \in V(G)} d_G(u,w_i) + |V(G)| (1 + \sum_{u \in V(H)} d_G(u,v)) + |V(H)| + 2|V(H)| + |V(G)||E(H)| + \sum_{w_i,w_j \in V(G)} d_G(w_i,w_j).
\]
\[
2|V(G)|\left(\left|\frac{|V(H)|}{2}\right|^2 - |E(H)| + 2\left|\frac{|V(G)|}{2}\right|^2\right) + |V(H)|^2 W(G) = \left(1 + 2|V(H)| + |V(H)|^2\right)W(G) + |V(G)|^2 |V(H)|^2 + 2|V(G)|\left(\frac{|V(H)|}{2} - |V(G)||E(H)| + 2|V(H)| + 2\left|\frac{|V(G)|}{2}\right|^2\right).\]
The result now follows.

**Example 1.3:**
In this example the Wiener index of $T_{n,3}$ (Fig. 3) and $B_n$ (Fig. 2), is computed. By Result 3 and

**Theorem 1.2:**
Implies that

$W(T_{n,3}) = \frac{n}{2}(3n^2 + 12n - 9)$, and $W(B_n) = \frac{n}{2}(3n^2 + 12n - 7)$.

**The Edge Szeged Index and Edge-Vertex Szeged Index of the Corona Product Graphs:**
In this section we compute the edge-vertex Szeged index and edge Szeged index of the corona product graph.

**Lemma 2.1:**
Let $H$ be free-triangle and regular graph. Then

(a) For every $e = u\nu \in E_1$, $n_u(e, G \circ H) = (|E(H)| + |V(H)|)n_u(e, G)$ and $n_v(e, G \circ H) = (|E(H)| + |V(H)|)n_v(e, G)$.

(b) For every $e = u\nu \in E_2$, $n_u(e, G \circ H) = d_H(u)$ and $n_v(e, G \circ H) = d_H(v)$.

(c) For every $e = u\nu \in E_3$, $n_u(e, G \circ H) = 1$ and $n_v(e, G \circ H) = |V(G \circ H)| - (d_H(u) + 1)$.

**Lemma 2.2:**
Let $H$ be free-triangle and regular graph. Then

(a) For every $e = u\nu \in E_1$, $m_u(e, G \circ H) = m_u(e, G) + (|E(H)| + |V(H)|)n_u(e, G)$ and $m_v(e, G \circ H) = m_v(e, G) + (|E(H)| + |V(H)|)n_v(e, G)$.

(b) For every $e = u\nu \in E_2$, $m_u(e, G \circ H) = m_u(e, H) + 1$ and $m_v(e, G \circ H) = m_v(e, H) + 1$.

(c) For every $e = u\nu \in E_3$, $m_u(e, G \circ H) = d_H(u)$ and $m_v(e, G \circ H) = |E(G \circ H)| - (d_H(u) + 1)$.

Now by Lemma 2.2 we have following theorem that will be computed the edge Szeged index of $G \circ H$.

**Theorem 2.3:**
Let $H$ be free-triangle and r-regular graph. Then

$S_{e_e}(G \circ H) = S_{e_e}(G) + 2(|V(H)| + |E(H)|)S_{e_v}(G) + (|V(H)| + |E(H)|)^2 S_e(G) + |V(G)|(|S_{e_e}(H) + P(H)) + 2|E(H)||V(G)|(|E(G \circ H)| - r - \frac{1}{2})$.

**Proof:**
Since $E_1$, $E_2$ and $E_3$ are disjoint, so $S_{e_e}(G \circ H) = \sum_{i=1}^{3} \sum_{e=uv \in E_i} m_u(e, G \circ H)m_v(e, G \circ H)$. On the other hand, by

Lemma 2.2(a),

$\sum_{e=uv \in E_1} m_u(e, G \circ H)m_v(e, G \circ H) =
\sum_{e \in E(G)} ([m_u(e, G) + (|V(H)| + |E(H)|)n_u(e, G)][m_v(e, G) + (|V(H)| + |E(H)|)n_v(e, G)]) =
\sum_{e \in E(G)} m_u(e, G)m_v(e, G) + \sum_{e \in E(G)} (|V(H)| + |E(H)|)[m_u(e, G)n_v(e, G) + m_v(e, G)n_u(e, G)] +
\sum_{e \in E(G)} (|V(H)| + |E(H)|)[n_u(e, G)n_v(e, G) = S_{e_e}(G) + 2(|V(H)| + |E(H)|)S_{e_v}(G) +
(|V(H)| + |E(H)|)^2 S_e(G)$. 

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Again by Lemma 2.2 (b):
\[
\sum_{e=uv \in E_2} m_u(e,G \circ H) m_v(e,G \circ H) = \sum_{e \in E_2} [m_u(e,H) + 1][m_v(e,G) + 1] =
\]
\[
|F(G)| \left( \sum_{i=1}^{|E(H)|} m_u(e,H) m_v(e,H) + \sum_{e \in E(H)} [m_u(e,H) + m_v(e,H)] \right) + \sum_{i=1}^{|E(H)|} \sum_{e \in E(H)} 1 =
\]
\[
|V(G)| (|Sz_e(H)| + PL(H)) + |V(G)| |E(H)|.
\]

Finally by Lemma 2.2(c) and this fact that \( r |V(H)| = 2 |E(H)| \) we obtain,
\[
\sum_{e=uv \in E_3} m_u(e,G \circ H) m_v(e,G \circ H) = \sum_{e \in E_3} r( |E(G \circ H)| - r - 1 ) = r |E(G \circ H)| |V(G)| |V(H)| -
\]
\[
r^2 |V(G)| |V(H)| - r |V(G)| |V(H)| = 2 |E(G \circ H)| |V(G)| |E(H)| -2r |V(G)| |E(H)| -2 |V(G)| |E(H)|.
\]

Therefore
\[
Sz_e(G \circ H) = Sz_e(G) + 2(|V(H)| + |E(H)|)Sz_ev(G) + (|V(H)| + |E(H)|)^2 Sz(G) +
\]
\[
|V(G)| (|Sz_e(H)| + PL(H)) + 2 |V(G)| |E(H)| (|E(G \circ H)| - r - \frac{1}{2}).
\]

The result now follows.

**Theorem 2.4:**

Let \( H \) be free-triangle and \( r \)-regular graph. Then
\[
Sz_e(G \circ H) = (1+ |V(H)|)Sz_ev(G) + (1+ |V(H)|) |E(H)| Sz_e(H) + |E(G \circ H)| |V(H)| |E(H)| -
\]
\[
2 |E(H)| |V(G)| - \frac{1}{2} |V(H)| |V(G)| + \frac{2}{2} |V(G)| |PL(H)| + \frac{1}{2} |E(G \circ H)| |V(G)| |V(H)|.
\]

**Proof:**

By definition, we have
\[
Sz_e(G \circ H) = \frac{3}{2} \sum_{i=1}^{|E_2|} \sum_{e=uv \in E_2} m_u(e,G \circ H) m_v(e,G \circ H) + m_v(e,G \circ H) n_u(e,G \circ H).
\]

By Lemmas 2.1(a) and 2.2 (a):
\[
\sum_{e=uv \in E_1} \sum_{e \in E_1} ([m_u(e,G) + (|V(H)| + |E(H)|)n_u(e,G)]([1 + |V(H)|]n_v(e,G)] +
\]
\[
2(|V(H)|) \sum_{e=uv \in E_2} m_u(e,G) m_v(e,G) m_u(e,G) n_v(e,G)] +
\]
\[
2(1+|V(H)|)(1+|E(H)|) \sum_{e=uv \in E_2} n_u(e,G) n_v(e,G) =
\]
\[
2(1+|V(H)|)Sz_ev(G) + 2(1+|V(H)|)(1+|E(H)|) Sz_e(G).
\]

Again by Lemmas 2.1(b) and 2.2 (b):
\[
\sum_{e=uv \in E_2} \sum_{e \in E_2} r(m_u(e,H) + m_v(e,H) n_u(e,H) + m_v(e,G \circ H)) =
\]
\[
[|F(G)| \sum_{i=1}^{|E(H)|} \sum_{e \in E(H)} r(m_u(e,H) + m_v(e,H)] + \sum_{e \in E_2} 2r = r |V(G)| |PL(H)| + 2r |V(G)| |E(H)|.
\]
Finally by Lemmas 2.1 and 2.2 (c):

\[
\sum_{e \in \text{nu} \in E_3} [m_e(e, G \circ H) n_e(e, G \circ H) + m_e(e, G \circ H) n_e(e, G \circ H)] = \\
\sum_{e \in E_3} [r V(G \circ H) - (r+1)] + |E(G \circ H)| - (r+1) = V(G) \| V(H) \| E(G \circ H) + r |V(G \circ H)| - (r+1)^2.
\]

Since \( r |V(H)| = 2 |E(H)| \), with easy calculations we obtain:

\[
S_{e_n}(G \circ H) = (1 + |V(H)|)S_{e_n}(G) + (1 + |V(H)|)|V(H)| E(H) S_{e_n}(G) + \frac{r}{2} |V(G)| |V(H)| + \frac{1}{2} |E(G \circ H)| V(G) \| V(H) \| E(H) - 2 |E(H)| V(G) \| V(H)| - \frac{1}{2} |V(H)| V(G) |.
\]

This proves the result.

We now apply our result to compute the above indices of some graphs. We first compute the following some topological indices for \( C_n \) and \( P_n \) where \( n \) is positive integer.

\[
S_{e_n}(P_n) = \left(\frac{n+1}{3}\right), \quad S_{e_n}(P_n) = \left(\frac{n+1}{3}\right) - (n-1)^2, \quad \text{and} \quad S_{e_{ev}}(P_n) = \left(\frac{n+1}{3}\right) - \left(\frac{n}{2}\right).
\]

\[
S_{e_n}(C_n) = \begin{cases} 
\frac{n^3}{4} & \text{n is even} \\
\frac{n(n-1)^2}{4} & \text{n is odd}
\end{cases}, \quad S_{e_{ev}}(C_n) = \begin{cases} 
\frac{n(n-2)^2}{4} & \text{n is even} \\
\frac{n(n-1)^2}{4} & \text{n is odd}
\end{cases}, \quad \text{PI}(C_n) = \begin{cases} 
n(n-2) & \text{n is even} \\
n(n-1) & \text{n is odd}
\end{cases}
\]

**Example 2.5:**

In this example the edge Szeged and edge-vertex Szeged index of \( T_{n,3} \) and \( B_n \) defined in Section 1, are computed. Therefore by Theorem 2.3.

\[
S_{e_n}(T_{n,3}) = \frac{4}{3} (2n^2 + 3n - 2) - 1, \quad \text{and} \quad S_{e_n}(B_n) = \frac{n}{2} (n^2 - 6n + 5) - 1.
\]

Also by Theorem 2.4:

\[
S_{e_{ev}}(T_{n,3}) = \frac{1}{2} (4n^2 + 11n - 9), \quad \text{and} \quad S_{e_{ev}}(B_n) = \frac{n}{2} (3n^2 + 3n - 4).
\]

**Example 2.6:**

In this example we computed the indices of \( S_{e_n}(P_n \circ C_m) \) and \( S_{e_{ev}}(P_n \circ C_m) \) for \( m \geq 4 \). By Theorems 3.3 and 3.5.

\[
S_{e_n}(J_{n,m+1}) = \frac{1}{12} \times \begin{cases} 
2n^3 - 36n^2 - 46n + 8n^3 m^2 + 8n^3 m + 48n^2 m^2 + n & \text{n is even} \\
24n^2 m + 3n^3 - 20nm^2 - 80nm - 12 & \\
2n^3 - 2n^2 + 24n + 8n^3 m^2 + 8n^3 m + 4n^2 m^2 + & \\
3n^3 - 14nm^2 + 15nm - 12 & \text{n is odd}
\end{cases},
\]

And

\[
S_{e_{ev}}(J_{n,m+1}) = \frac{1}{6} \times \begin{cases} 
n^3 + 9n^2 - 58n + 2n^3 m^2 + 3n^3 m + 15n^2 m - nm^2 + 4nm & \text{n is even} \\
n^3 + 9n^2 - 58n + 2n^3 m^2 + 3n^3 m + 15n^2 m - nm^2 + 5nm & \text{n is odd}
\end{cases}
\]
The PI Index Vertex PI Index of the Corona Product Graphs:
In this section we compute the PI index and vertex PI index of the corona product graph.

Theorem 4.1:
Let $H$ be free-triangle and $r$-regular graph. Then

$$\pi(G \circ H) = \pi(G) + |V(G)| \pi(H) + \pi(V(H)) |E(H)| + |E(G \circ H)| |V(G)| |E(H)|.$$  

Proof:
Since $E_1, E_2$ and $E_3$ are disjoint, $\pi(G \circ H) = \sum_{i=1}^{3} \sum_{e=uv \in E_i} \left( m_{e}(G \circ H) + m_{v}(G \circ H) \right)$.

On the other hand, by Lemma 3.2(a),

$$\pi(G \circ H) = \sum_{e \in E(G)} \left( m_{e}(G \circ H) + |V(H)| + |E(H)| n_{e}(e, G) \right) + \sum_{e \in E(G)} \left( m_{v}(e, G) + |V(H)| + |E(H)| n_{v}(e, G) \right).$$

Again by Lemma 3.2(b):

$$\pi(G \circ H) = \sum_{e=uv \in E_2} \sum_{V(G)} \left( m_{e}(e, G) + m_{v}(e, G) \right) + \sum_{V(G)} \left( |V(G)| + |E(H)| \right) \pi(H) + 2 |V(G)| |E(H)|.$$

Finally by Lemma 3.2(c) we obtain

$$\pi(G \circ H) = \sum_{e \in E_3} (r+ |E(G \circ H)| - r - 1) =$$

$$|E(G \circ H)| |V(G)| |V(H)| - |V(G)| |V(H)|.$$

Therefore

$$\pi(G \circ H) = \pi(G) + |V(G)| \pi(H) + (|V(H)| + |E(H)|) \pi(G) + |V(G)| |E(H)| + |E(G \circ H)| |V(G)| |E(H)|.$$  

The result now follows:

Example 4.3
One can see that $\pi(P_n) = (n-1)(n-2)$ and $\pi_v(P_n) = n(n-1)$, so

$$\pi(T_{n,3}) = \pi(P_n \circ K_2) = 12n^2 - 8n + 2, \quad \pi(B_n) = \pi(P_n \circ K_2) = 9n^2 - 9n + 2,$$  

and one can see that $\pi_v(T_{n,3}) = \pi_v(B_n) = 3n(3n - 1)$.

Example 4.4:
For path graph and cycle graph where $m \geq 4$,

$$\pi(J_{n,m+1}) = \begin{cases} 2n^2 + 3n^2 m + nm^2 - 4nm + n^2 - 3n + 2 & n \text{ is even} \\ 2n^2 + 3n^2 m + nm^2 - 3nm + n^2 - 3n + 2 & n \text{ is odd} \end{cases}$$
and

\[ PL_v(J_{n,m+1}) = n^2m^2 + 2n^2m + nm + n^2 - n. \]

REFERENCES