Numerical Solution of the Two Point Boundary Value Problems By Using Wavelet Bases of Hermite Cubic Spline Wavelets

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Abstract: In this paper, compactly supported Hermite cubic spline wavelets are developed to approximate the solutions of the linear two point boundary value problems. These wavelets constructed and properties of these wavelets utilized to reduce the computation of integral equations to some algebraic equations. The corresponding stiffness matrix arises from the discretization of the problem is sparse and the condition number of this matrix is uniformly bounded.

Key words: Two Point Boundary Value Problem, Wavelet, Approximation, Hermite cubic spline Wavelet bases.

INTRODUCTION

Wavelets theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering disciplines, particularly; wavelets are very successfully used in signal analysis for waveform representations and segmentations, time frequency analysis and fast algorithms for easy implementation. Wavelet numerical method is suited to applications such as (Canuto, C., A. Tabacco and K. Urban, 1999; Cohen, A., 2003). Several numerical methods for approximating the solution of integral and differential equations are known. In recent years, the application of methods based on spline wavelets have influence on many areas of applied mathematics. Spline wavelets are used as a powerful tool in the numerical analysis of differential and integral equations (Lakestani, M., et al., 2005; Maleknejad, K., et al., 2010).

In present paper, we apply compactly supported Hermite cubic spline wavelets to solve the two point boundary value problems. The paper is organized as follows. In section 2, we describe the formulation of the spline scaling functions and wavelets required for our subsequent development. In sections 3 and 4 we will apply the wavelets for numerical solutions of the two point boundary value problems with the Dirichlet boundary condition. The corresponding stiffness matrix arises from the discretization of the problem is sparse and the condition number of this matrix is uniformly bounded (independent of the number of bases functions) and finally in section 5, we report our numerical finding and computational results demonstrate the accuracy of the proposed numerical scheme.

Wavelets of Hermite Cubic Splines on The Interval:

Let \( \psi (x) \) and \( \phi (x) \) be the cubic splines given by

\[
\phi (x) = \begin{cases} 
3x^2 - 2x^3, & x \in [0,1], \\
3(2-x)^3 - 2(2-x)^2, & x \in [1,2], \\
0, & x \in [-0,2],
\end{cases}
\]

and

\[
\phi (x) = \begin{cases} 
x^3 - x^2, & x \in [0,1], \\
(2-x)^3 - (2-x)^2, & x \in [1,2], \\
0, & x \in [-0,2];
\end{cases}
\]

In [7], two wavelets \( \psi_1 (x) \) and \( \psi_2 (x) \) are introduced as follows:

\[
\psi_1 (x) = -2\phi (2x) + 4\phi (x+1) - 2\phi (x+2) + 21\phi (2x) - 21\phi (2x+2),
\]

\[
\psi_2 (x) = -\phi (2x) + \phi (x+2) + 9\phi (2x) + 12\phi (2x+1) + 9\phi (2x+2).
\]
These wavelets can be easily adapted to the interval \([0,1]\). For reader's convenience, we briefly recall the basic ideas. By \(L_2(0,1)\) we denote the space of all square integrable real valued functions on \((0,1)\). The inner product in \(L_2(0,1)\) is defined as

\[
\langle u, v \rangle = \int_0^1 u(x)v(x)\,dx, \quad u, v \in L_2(0,1).
\]

Let \(H^1(0,1)\) be the space of all functions \(u\) in \(L_2(0,1)\) for which \(u' \in L_2(0,1)\). Let \(H_0^1(0,1)\) be the closure of the set \(\{u \in C[0,1] \cap C^1(0,1) : u(0) = u(1) = 0\}\) in the space \(H^1(0,1)\). \(H_0^1(0,1)\) has the following decomposition \(H_0^1(0,1) = V_0 \oplus W_1 \oplus W_2 \oplus \cdots\), where \(W_n\) is the linear span of

\[
\Psi_n = \{\psi_n(2^n \cdot - j) : j = 1, \ldots, 2^n - 1\} \cup \{\psi_n(2^n \cdot - j)_{(0,1)} : j = 0, \ldots, 2^n\},
\]

And

\[
\Phi_n = \{\phi_n(2^n \cdot - j) : j = 0, \ldots, 2^n - 2\} \cup \{\phi_n(2^n \cdot - j)_{(0,1)} : j = -1, \ldots, 2^n - 1\},
\]

is a basis for \(V_n\). For \(n = 1, 2, \ldots\) and \(x \in (0,1)\), let

\[
\psi_{n,j}(x) = \frac{2^{-n}}{\sqrt{729.6}} \psi_n\left(2^n x - \frac{j}{2}\right), \quad (j = 2, 4, \ldots, 2^n - 2),
\]

\[
\psi_{n,j}(x) = \frac{2^{-n}}{\sqrt{153.6}} \psi_n\left(2^n x - \frac{j-1}{2}\right)_{(0,1)}, \quad (j = 3, 5, \ldots, 2^n - 1),
\]

\[
\psi_{n,1}(x) = \psi_2(2^n x)_{(0,1)},
\]

\[
\psi_{n,2^n}(x) = \psi_2(2^n x - 2^n)_{(0,1)}.
\]

For \(x \in (0,1)\) let

\[
\phi_{1}(x) = \frac{5}{24} \phi(x),
\]

\[
\phi_{2}(x) = \frac{15}{4} \phi(2x+1)_{(0,1)},
\]

\[
\phi_{3}(x) = \frac{15}{8} \phi(2x)_{(0,1)},
\]

\[
\phi_{4}(x) = \frac{15}{4} \phi(2x-1)_{(0,1)}.
\]

Clearly, \(H_0^1(0,1)\) is spanned by

\[
\{\phi_{1,j}(x), j = 1, 2, 3, 4\} \cup \{\psi_{n,j}(x), n = 1, 2, \ldots\} \cup \{\psi_{n,j}(x), n = 1, 2, \ldots, 2^n\}.
\]

**Two Point Boundary Value Problem:**

Consider the second order equation

\[
v''(x) = f(x, v(x), v'(x)), \quad x \in [0,1],
\]

with the initial conditions
Equations (1) and (2) define a two point boundary value problem [8]. Note that the restriction of the interval into the interval $[0,1]$ in the relation (1) has no loss of generality, since a problem on any finite interval may be converted to the one on the interval $[0,1]$. (See remark 3.1).

**Remark 3.1:**

Consider the boundary value problem $u''(z) = g(z, u(z), u'(z))$, with $u(a) = \alpha$ and $u(b) = \beta$. By the change of variable $t = (b-a)x + a$, this problem is equivalent to (1) and (2) with $v(x) = u((b-a)x + a)$ and $f(x, v(x), v'(x)) = (b-a)^2 g((b-a)x + a, v(x), (b-a)^{-1}v'(x))$.

In this section we treat only linear problems, in which the equation (1) may be written in the form

$$v''(x) = b(x)v'(x) + c(x)v(x) + d(x), \quad x \in [0,1],$$

where $b$, $c$ and $d$ are given functions. The boundary conditions that we consider first are the conditions given in (2). Later, we will treat other types of the boundary conditions. Equations (3) and (2) define a linear two point boundary value problem for the unknown function $v$ and our task is to develop procedures to approximate the solution. We assume that the problem has a unique solution that is the result of being at least two times continuously differentiable of the unknown function.

We consider the special case of (3) in which $b(x) \equiv 0$, so that the equation is

$$v''(x) = c(x)v(x) + d(x), \quad x \in [0,1].$$

Assume that $c(x) \geq 0$ for $x \in [0,1]$. This is a sufficient condition for the problem (2) and (4) to have a unique solution.

Consider the following linear two point boundary value problem

$$-v''(x) + q(x)v = f(x), \quad x \in [0,1],$$

with the conditions

$$v(0) = 0, \quad v(1) = 0,$$

where, for simplicity, we have taken the interval to be $[0,1]$ and the boundary conditions to be zero (see remark 3.2).

**Remark 3.2:**

The boundary value problem

$$v''(x) + p(x)v'(x) + q(x)v(x) = f(x), \quad v(0) = \alpha, v(1) = \beta,$$

can be converted to a problem with zero boundary conditions as follows.

Let $u(x) = v(x) - (\beta - \alpha)x - \alpha$, then $u(0) = u(1) = 0$ and $u$ satisfies the differential equation

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x) - \alpha q(x) - \left[p(x) + q(x)\right](\beta - \alpha).$$

Assume that there exist finite constant $C_1$ such that

$$0 \leq q(x) \leq C_1, \quad x \in [0,1],$$

by a result in theory of ODE, there is a unique function $v$ satisfying equation (5) and the boundary conditions (6).

*The Galerkin Method to Solve the Linear Two Point Boundary Value Problem by Using Wavelet Bases of*
Hermite Cubic Splines:

A function \( v(x) \) defined over \([0,1]\) can be represented by Hermite cubic spline wavelets as

\[
v(x) = \sum_{i=0}^{1} c_i \phi_i + \sum_{j=1}^{n} \sum_{i=1}^{4} d_{ij} \psi_{ij},
\]

where \( \phi_i \) and \( \psi_{ij} \) are scaling and wavelet functions, respectively. If the infinite series in (7) is truncated, then (7) can be written as

\[
v(x) \square u(x) = \sum_{i=0}^{1} c_i \phi_i + \sum_{j=1}^{n} \sum_{i=1}^{4} d_{ij} \psi_{ij} = \sum_{j=1}^{n} c_j g_j(x),
\]

where the basis functions \( g_j \) satisfy the boundary conditions

\[
g_j(0) = g_j(1) = 0, \quad j = 1, \cdots, n.
\]

If (9) holds, then the approximate solution \( u \) given by (8) satisfies the boundary conditions. We use the wavelet set \( G_n = \{ g_{1n}, \cdots, g_{2n+1}, \psi_{1n}, \cdots, \psi_{2n+1} \} \) as a basis for \( V_n \), where \( g_j = \phi_{i,j} \) for \( j = 1, \cdots, 4 \) and \( g_{2n+1,j} = \psi_{i,j} \) for \( n=1, 2, \cdots \) and \( j = 1, \cdots, 2n+1 \).

Let the residual function for \( u(x) \) be defined by

\[
r(x) = -u''(x) + q(x)u(x) - f(x), \quad x \in [0,1].
\]

If \( u(x) \) were the exact solution of (5), then the residual function would be identically zero. Obviously, the residual is orthogonal to every function and, in particular, it is orthogonal to the set of basis functions. However, we can’t expect \( u(x) \) to be the exact solution because we restrict \( u(x) \) to be a linear combination of the basis functions. By the Galerkin method the residual function is orthogonal to all of the basis functions \( g_{1n}, \cdots, g_{2n+1} \)

\[
\int_{0}^{1} [-u''(x) + q(x)u(x) - f(x)] g_i(x) dx = 0, \quad i = 1, \cdots, 2n+1.
\]

If we put (8) into (10) and interchange the summation and integration, we obtain

\[
\sum_{j=1}^{n} c_j \int_{0}^{1} [-g_j''(x) + q(x)g_j(x)] g_i(x) dx = \int_{0}^{1} f(x) g_i(x) dx, \quad i = 1, \cdots, 2n+1.
\]

This leads to a system of linear equations of the form \( AC = F \) with

\[
C = \begin{bmatrix} c_1, c_2, \cdots, c_{2n+1} \end{bmatrix}^T,
\]

\[
F = \begin{bmatrix} f_1, f_2, \cdots, f_{2n+1} \end{bmatrix}^T, \quad \text{where} \quad f_i = \int_{0}^{1} f(x) g_i(x) dx,
\]

\[
A = \begin{bmatrix} a_y \end{bmatrix} = \begin{bmatrix} \int_{0}^{1} [-g_j''(x) + q(x)g_j(x)] g_i(x) dx \end{bmatrix}.
\]

If we integrate the first term in this integral by parts,

\[
\int_{0}^{1} g_j''(x) g_i(x) dx = g_j'(x) g_i(x) \bigg|_{0}^{1} - \int_{0}^{1} g_j'(x) g_i'(x) dx,
\]

and note that the first term vanishes because \( g_i \) is zero at the end points, we can rewrite \( a_y \) as

\[
a_y = \int_{0}^{1} g_j'(x) g_i'(x) dx + \int_{0}^{1} q(x) g_j(x) g_i(x) dx.
\]

Let \( L \) be an operator defined as

\[
Lu(t) = -u''(t) + q(t)u(t), \quad u(t) \in L_2(0,1),
\]

then,
\( a_j = \langle Lg_j(x), g(x) \rangle = \int_0^1 g_j'(x) g'(x) dx + \int_0^1 g(x) g_j'(x) g(x) dx, \quad j = 1, 2, \ldots, 2^n. \)

**Theorem 4.1:**
The sequence \( \{g_j'(x)\}_{k=1,2,\ldots} \) is a Riesz sequence in \( L_2(0,1) \).

**Proof:**
(See [7]).

**Corollary 4.1:**
For every \( g(t) = \sum_{j=1}^{\infty} b_j \phi_j(t) + \sum_{n=1}^{\infty} \sum_{j=1}^{2^n} b_{n,j} \psi_{n,j}(t) \in L_2(0,1) \), there exist two positive constants \( C_2 \) and \( C_3 \) such that
\[
C_2 A \leq \int_0^t |g'(t)|^2 dt \leq C_3 A,
\]
where
\[
A = \left( \sum_{j=1}^{\infty} |c_j|^2 + \sum_{n=1}^{\infty} \sum_{j=1}^{2^n} |b_{n,j}|^2 \right).
\]

**Theorem 4.2:**
Let \( L \) be an operator defined as \( Lu(t) = -u''(t) + q(t)u(t) \), \( u(t) \in L_2(0,1) \), satisfying
\[
\exists C_i (\text{constant}): 0 \leq q(t) \leq C_i, \quad t \in [0,1],
\]
and \( g \in L_1(0,1) \) satisfies \( g(0) = g(1) = 0 \), then
\[
C_2 A \leq \langle Lg, g \rangle \leq (1 + C_i) C_3 A,
\]
where \( A \) is defined in (14).

**Proof:**
Note that by integration by parts,
\[
\langle Lg(t), g(t) \rangle = -\int_0^1 g''(t) g(t) dt + \int_0^1 g(t) g'(t)^2 dt = \int_0^1 (g'(t))^2 dt + \int_0^1 g(t) g'(t)^2 dt,
\]
and
\[
\int_0^1 |g'(t)|^2 dt \leq \int_0^1 g'(t) g'(t) dt = \langle g'(t), g'(t) \rangle,
\]
also,
\[
0 \leq \int_0^1 g(t) |g(t)|^2 dt = \langle q(t) g(t), g(t) \rangle,
\]
by adding these two inequalities,
\[
\int_0^1 |g'(t)|^2 dt \leq \langle Lg, g \rangle.
\]
Also, note that
\[
\langle g'(t), g'(t) \rangle \leq \int_0^1 |g'(t)|^2 dt.
\]

Because \( g(0) = 0 \) then \( g(t) = \int_0^t g'(s) ds \). Hence by the Cauchy–Schwartz inequality
\[
|g(t)|^2 \leq (\int_0^1 |g'(s)|^2 ds)(\int_0^t |g'(s)|^2 ds) \leq (\int_0^1 |g'(s)|^2 ds) \int_0^t |g'(s)|^2 ds, \quad \forall t \in [0,1],
\]
therefore,
\[
\int_0^t |g(t)|^2 \, dt \leq (\int_0^t |g'(s)|^2 \, ds)^{\frac{1}{2}} (\int_0^t ds)^{\frac{1}{2}} = \int_0^t |g'(s)|^2 \, ds.
\]

Hence,
\[
\langle q(t) g(t), g(i) \rangle = \int_0^i |g(t)|^2 \, dt \leq C \int_0^t |g'(t)|^2 \, dt. \tag{17}
\]

by adding (16) and (17), we have
\[
\langle Lg, g \rangle \leq (1 + C_i) \int_0^t |g'(t)|^2 \, dt. \tag{18}
\]

The proof of theorem is completed by the relations (13), (15) and (18).

**Corollary 4.2:**

*The condition number of the stiffness matrix $A$ is uniformly bounded (independent of $n$).*

Let us recall that the condition number of an invertible square matrix $A$ is defined by

\[
\text{cond}(A) = \frac{\|A\| \|A^{-1}\|}{\|I\|},
\]

where $\|\|$ is a matrix norm. The spectral condition number of $A$ is defined as

\[
\frac{\|z\|_{\text{max}}}{\|z\|_{\text{min}}},
\]

where $z$ is a matrix norm. The spectral condition number of $A$ is defined as

\[
\|z\|_{\text{max}} = \max \|z\| : \lambda \text{ is an eigenvalue of } A,
\]

and
\[
\|z\|_{\text{min}} = \min \|z\| : \lambda \text{ is an eigenvalue of } A.
\]

If $A$ is a (real) symmetric matrix, then its condition number with respect to the 2-norm is equal to its spectral condition number (Ciarlet, P.G., 1989). The condition number of a matrix $A$ measures the stability of the linear system $AC = F$ under perturbations of $F$. Think of perturbing $F$ by $\delta F$ to obtain $F + \delta F$. Let $C$ be the solution of $AC = F$ and $\delta C$ be the solution of $A\delta C = \delta F$, then by linearity, $AC + \delta C = F + \delta F$. The stability of the linear system is most naturally described by comparing the relative size $\|\delta C\|/\|C\|$ of the change in the solution to the relative size $\|\delta F\|/\|F\|$ of the change in the given data. The condition number is the maximum value of this ratio.

**Theorem 4.3:**

*Let $A$ is an invertible matrix, $C \neq 0$, $AC = F$ and $A\delta C = \delta F$ then*

\[
\frac{\|\delta C\|}{\|C\|} \leq \text{cond}(A) \frac{\|\delta F\|}{\|F\|}. \tag{19}
\]

Moreover, $\text{cond}(A)$ cannot be replaced by any smaller number in relation (19).

**Test Problems:**

In order to verify the efficiency of the method introduced in the previous section, the results of several numerical experiments are presented and compared with either exact solutions. In the following tables we applied the norm

\[
\|v(x) - u(x)\| = \left( \int_0^1 (v(x) - u(x))^2 \, dt \right)^{\frac{1}{2}}\]

The programs are compiled in *Maple 12.*

**Example 1:**

Consider the problem

\[
\begin{aligned}
-v''(x) &= \pi^2 \sin(\pi x), \quad x \in [0,1], \\
v(0) &= 0, \quad v(1) = 0.
\end{aligned}
\]
The exact solution of this problem is \( v(x) = \sin(\pi x) \). Table 1 shows the absolute values of errors of the approximation solutions. In this example, stiffness matrix \( A = \left( (g_i^\prime, g_j^\prime) \right)_{i,j=1,\ldots,2^n+1} \) is block diagonal and each block is a banded matrix. By corollary 4.2, the condition number of the matrix \( A \) is uniformly bounded. In table 2, the condition number of \( A \) for \( n = 6,\cdots,12 \) are shown.

**Table 1:** The error in \( v' \) for example 1.

<table>
<thead>
<tr>
<th>( 2^n+1 )</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>v'(x) - u(x)</td>
<td>)</td>
<td>2.388\times10^{-3}</td>
<td>2.092\times10^{-4}</td>
<td>1.467\times10^{-5}</td>
</tr>
</tbody>
</table>

**Table 2:** Condition number of the stiffness matrix \( A \) for example 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{cond}(A) )</td>
<td>3.7396</td>
<td>3.7474</td>
<td>3.7494</td>
<td>3.7798</td>
<td>3.7798</td>
<td>3.7798</td>
<td>3.7798</td>
</tr>
</tbody>
</table>

**Example 2:**

Consider the linear two point boundary value problem

\[
\begin{align*}
-v''(x) + v(x) &= \left( -2 + x(x-1)(\pi^2 + 1) \right) \sin(\pi x) + 2\pi (1-2x)\cos(\pi x), \\
v(0) &= 0, \quad v(1) = 0.
\end{align*}
\]

The exact solution of this problem is \( v(x) = x(x-1)\sin(\pi x) \). Table 3 shows the absolute values of errors of the approximation solutions. The stiffness matrix \( A = \left( (g_i^\prime, g_j^\prime) + (g_i, g_j) \right)_{i,j=1,\ldots,2^n+1} \) is still a sparse matrix and the condition number of \( A \) is uniformly bounded (see Table 4).

**Table 3:** The error in \( v \) for example 2.

<table>
<thead>
<tr>
<th>( 2^n+1 )</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>v(x) - u(x)</td>
<td>)</td>
<td>2.038\times10^{-3}</td>
<td>2.2452\times10^{-4}</td>
<td>1.7123\times10^{-5}</td>
</tr>
</tbody>
</table>

**Table 4:** Condition number of the stiffness matrix \( A \) for example 2.

<table>
<thead>
<tr>
<th>( n )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{cond}(A) )</td>
<td>4.628</td>
<td>4.719</td>
<td>4.784</td>
<td>4.831</td>
<td>4.864</td>
<td>4.890</td>
<td>4.907</td>
</tr>
</tbody>
</table>

**Conclusion:**

In the present work, a technique has been developed for solving some two point boundary value problems. The method is based on Hermite cubic spline wavelets. The problem has been reduced for solving a system of linear algebraic equations and the stiffness matrix of the system is sparse and condition number of this matrix is uniformly bounded (independent of the number of bases functions). The computational results demonstrate the advantage of this wavelet basis.

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