Estimation of Diffusion Coefficient in Gas Exchange Process with in Human Respiration Via an Inverse Problem

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Abstract: This paper is intended to provide a stochastic approach involving the combined use of the Feynman-Kac formula and Monte Carlo method as a solution algorithm for estimating the time-dependent effective diffusivity in a one-dimensional parabolic inverse problem. The inverse problem is purposed to design a mathematical model for the gas-diffusion process within the alveolar region of the human's lung. The model depends on a representative physical property of the alveolar region termed the effective diffusivity. In the present study, the functional form of the effective diffusivity is unknown a priori. The unknown effective diffusivity is approximated by the polynomial form. To modify the coefficients of the polynomial form of the unknown effective diffusivity, we introduce a deterministic optimization method based on least squares minimization. A numerical test is performed in order to show the efficiency and accuracy of the present work.

Key words: Gas-exchange process, Human respiration, Inverse problem, Feynman-Kac formula, Monte Carlo method.

INTRODUCTION

In the present work a specific stochastic combined algorithm is used for obtaining the solution of inverse diffusion equation as part of a parabolic inverse problem that arise in gas-diffusion process within the alveolar region of the lung during human respiration. This algorithm uses Feynman-Kac formula to represent the solution of the parabolic inverse problem at a point as the expected value of functionals of Brownian motion trajectories started at the point of interest. We are interested to have solution of a parabolic inverse problem without any need to discretize the problem domain. For many problems described by partial differential equations (PDEs) such solutions are delivered by the so-called Feynman-Kac formulas (Csaki, E., 1993; Modeste, N., 2006; Budaev, B.V. and D.B. Bogy, 2003). The literature reviews showed that E. Csaki (1993) have applied Feynman-Kac formula to solve initial-boundary value problems. In (Csaki, E., 1993) a discrete Feynman-Kac formula has employed for linear parabolic PDEs with zero boundary conditions. Modeste et al. (2006) have used a kind of nonlinear Feynman-Kac formula to give a probabilistic interpretation of the solutions of parabolic quasi-linear PDEs. Budaev and Bogy (2003) presented a probabilistic approach to systems of PDEs on the basis of the well-known Feynman-Kac formula. To date various methods have been developed for the analysis of the parabolic inverse problems involving the estimation of boundary condition or diffusion coefficient from measurement inside the material (Shidfar, A., 2009; Wang, J. and N. Zabaras, 2004; Shidfar, A., 2007; Shidfar, A., 2006; Shidfar, A., 2006; Farnoosh, R. and M. Ebrahimi, 2010; Dehghan, M., 2005). Shidfar et al. (2009) have applied an algorithm based on conjugate gradient method to estimate the unknown time dependent melt depth during laser material processing in liquid phase. In this article the determination of the melt depth is treated as a one-dimensional, transient, inverse heat conduction problem. Numerical procedure shows a good agreement with experimental and analytical results. In (Wang, J. and N. Zabaras, 2004), Wang and Zabaras have used a Bayesian inference approach to the inverse heat conduction problem. Their work is based on Monte Carlo method and experiment results show a good estimation on the linear inverse problems in two dimensions. Shidfar et al. (2007) have used an accurate and stable numerical algorithm based on finite difference method to solve an inverse parabolic problem in one dimension.

To our best knowledge the problem of gas-exchange in human respiratory system with unknown time dependent diffusion coefficient has not been studied. Furthermore, according to latest information from the research works it is believed that the solution of inverse problem based on stochastic algorithm included the Feynman-Kac formula has been investigated for the first time in the present study.

Description of the Problem:

Formulation of the direct and inverse problem is given as follows:

1) Direct Problem:
The mathematical formulation of a one dimensional linear parabolic problem is given as follows:
Where $t_f$ is the final time for measurements. The direct problem considered here is concerned with the determination of the medium partial pressure when the time-dependent diffusion coefficient $a(t)$, the initial condition $f(x)$ and the boundary conditions $\psi(t)$ and $\xi(t)$ are known continuous functions.

**Theorem 1:**

The problem (1)-(4) has a unique solution if $f(x)$, $\psi(t)$ and $\xi(t)$ are continuous functions and $a(t) > 0$, (Cannon, J.R., 1984).

### II) Inverse Problem:

For the inverse problem, the diffusion coefficient $a(t)$ is regarded as being unknown. In addition, an overspecified condition is also considered available. To estimate the unknown coefficient $a(t)$, the additional information of measurements on the boundary $x = 1$, $0 < x < 1$, is required. Let the partial pressure measurements taken at $x = x_i$ over the time period $t_f$ be denoted by

$$u(x_i, t) = h(t), \quad 0 < t < t_f,$$

The additional condition is performed based on simulating numerically the gas-diffusion process during a single-breath, lung-diffusing capacity test, routinely performed in clinical settings (Kulish, V., 2006).

It is evident that for an unknown function $a(t)$, the problem (1)-(4) is under-determined and we are forced to impose additional information (5) to provide a unique solution pair $(u, a(t))$ to the inverse problem (1)-(5). We note that the measured partial pressure $u(x_i, t) = h(t)$ should contain measurement errors. Therefore the inverse problem can be stated as follows:

by utilizing the above-mentioned measured data, estimate the unknown function $a(t)$ over the entire space and time domain.

It is worth noting that $a(t) > 0$ (Cannon, J.R., 1984). Certain types of physical problems can be modeled by (1)-(5). The coefficient $a(t)$ can represent physical quantities, for example, the conductivity of a medium. The existence and uniqueness of the solutions to this problem and also some more applications are discussed in (Dehghan, M., 2005; Cannon, J.R., 1984). The numerical solution of the problem (1)-(5) has been discussed by several authors. For example it can be found that the numerical based on several finite difference schemes, proposed by Dehghan (2005), applied to the above mentioned parabolic inverse problem (1)-(5). The results show that the accuracy of this work are very reasonable.

### Overview of the Solution Algorithm:

The application of the present method to find the solution of problem (1)-(5) can be divided into the following steps:

**Step 1. Remove the Time Dependent Diffusion Coefficient:**

We utilize the transformation

$$a(t) = \frac{2}{t} \int_0^t a(y) dy, \quad 0 \leq t \leq t_f,$$

to reduce the equation (1) to that involving the diffusion equation with constant diffusion coefficient. Since

\[
\begin{align*}
    u_t &= a(t)u_{xx}, \quad 0 < x < 1, \quad 0 < t < t_f, \\
    u(x,0) &= f(x), \quad 0 < x < 1, \\
    u(0,t) &= \psi(t), \quad 0 < t < t_f, \\
    u(1,t) &= \xi(t), \quad 0 < t < t_f,
\end{align*}
\]
\( \theta'(t) = 2a(t) > 0, \quad 0 \leq t \leq t_f, \)

then there exists a unique function \( \varphi(\theta) \) such that

\[
\theta(\varphi(\tau)) = \tau, \quad 0 \leq \tau \leq \theta(t_f),
\]

\[
\varphi(\theta(t_i)) = t, \quad 0 \leq t \leq t_f,
\]

and

\[
\varphi'(\tau) = \theta'(\varphi(\tau))^{-1} = (2a(\varphi(\tau)))^{-1} = (2a(t))^{-1}, \quad 0 \leq \tau \leq \theta(t_f).
\]

Let

\[ U(x, \eta) = u(x, \varphi(\eta)), \]

then

\[
U'(x, \eta) = u_1(x, \varphi(\eta)) \varphi' (\eta) = u_1(x, \varphi(\eta)) (a(\varphi(\eta)))^{-1} = \frac{1}{2} u_\eta(x, \varphi(\eta)) = \frac{1}{2} U_\eta(x, \eta).
\]

Consequently, to obtain the representation for \( u(x, t) \), we substitute \( \eta = \theta(t) \) into the representation for \( U(x, \eta) \). Therefore the problem (1)-(5) becomes

\[
U'(x, \eta) = \frac{1}{2} U_\eta, \quad 0 < x < 1, \quad 0 < \eta \leq t_f,
\]

(6)

\[ U(x, 0) = f(x), \quad 0 < x < 1, \]

(7)

\[ U(0, \eta) = \psi(\varphi(\eta)), \quad 0 < \eta \leq T_f, \]

(8)

\[ U(1, \eta) = \xi(\varphi(\eta)), \quad 0 < \eta \leq T_f, \]

(9)

\[ U(x, \eta) = h(\varphi(\eta)), \quad 0 < \eta \leq T_f \]

(10)

where \( T_f = \theta(t_f) \).

**Step 2. Feynman-Kac Formula:**

In this step we begin with a method to solve PDEs based on the representation of point solutions of the PDEs as expected values of functionals of stochastic processes defined by the Feynman-Kac formula. The particular stochastic processes that arise in the Feynman-Kac formula are solutions to specific stochastic differential equations defined by the characteristics of the differential operator in the PDE. The Feynman-Kac formula is applicable to a wide class of linear initial and initial-boundary value problems for elliptic and parabolic PDEs. On the basis of the well-known Feynman-Kac formula providing explicit probabilistic representation for the solution of problem (6)-(10).

**Theorem 2. (The Feynman-Kac formula):**

Let \( f : R \rightarrow R \) be a continuous differentiable function with compact support in \( R \) and \( W(x, \eta) : R \times R \rightarrow R \) is a function with continuous derivatives up to order 1 and 2 with respect to \( \eta \) and \( x \), respectively.

a) Put

\[
U(x, \eta) = E[f(X_{\eta})],
\]

(11)
then $U(x, \eta)$ satisfies the problem (6)-(9).

b) Moreover, if $W(x, \eta)$ is bounded on $K \times R$ for each compact $K \subseteq R$ and $W$ solves (6)-(9), then $W(x, \eta) = U(x, \eta)$, given by (11).

For proof of the Theorem 2 we refer to (Oksendal, B., 1998). Now we use the Feynman-Kac formula to obtain the solution $U(x, \eta)$ of the direct problem (6)-(9), given previously by using an approximated $\hat{a}(t)$ for the exact $a(t)$. Therefore $U(x, \eta)$ is given by (11).

For some cases the expectation value of random variable $Y = f(X_{\eta})$ can be calculate with mathematical softwares such as Mathematica 7, exactly. In fact, based on what the initial condition $u(x,0) = f(x)$ is, we can obtain the solution of direct problem (1)-(4) exactly or approximately. For example when we consider $f(x) = x^2$ the exact solution of $E[f(X_{\eta})]$ can be compute, simply. For other cases such as $f(x) = \sin x$, the Monte Carlo method is employed to solve the integral $E[f(X_{\eta})] = \frac{1}{\sqrt{2\pi\eta}} \int_{-\infty}^{\infty} (\sin \eta \exp \frac{-(x-y)^2}{2\eta})dy$.

**Step 3. Monte Carlo Simulation to Estimate $E[f(X_{\eta})]$**

The idea of Monte Carlo simulation is to draw an identical independently distributed set of samples $\{X^{(i)}_{\eta}\}_{i=1}^N$ from a target prior probability density function $p(X_{\eta})$. These $N$ samples can be used to approximate the target density with the following empirical point-mass function:

$$p_N(X_{\eta}) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{(i)}_{\eta}}(X_{\eta}),$$

where $\delta_{X^{(i)}_{\eta}}(X_{\eta})$ denotes the delta-Dirac mass located at $X^{(i)}_{\eta}$. Consequently, one can approximate the expectation of any function $f$ of $X_{\eta}$ by its mean as follows:

$$E_N[f(X_{\eta})] = \frac{1}{N} \sum_{i=1}^{N} f(X^{(i)}_{\eta}).$$

By the strong law of large numbers, $E_N(f)$ converges to $E(f)$, i.e.

$$\lim_{N \to \infty} E_N[f(X_{\eta})] = E[f(X_{\eta})] = \int f(x) p(x) dx.$$

**Step 4. Optimization Technique**

In this work the polynomial form is proposed for the unknown function $a(t)$ before performing the inverse calculation. Therefore $a(t)$ can be approximated as the following forms:

1) A polynomial with degree $m$ and real coefficients:

$$\hat{a}(t) = c_0 + c_1 t + \ldots + c_m t^m.$$

2) A trigonometric polynomial with degree $m$ and real coefficients:

$$\hat{a}(t) = c_0 + \sum_{j=1}^{m} c_{2j-1} \cos(jt) + \sum_{j=1}^{m} c_{2j} \sin(jt),$$

where $\{c_0, c_1, \ldots, c_m\}$ are constants which remain to be determined simultaneously. The unknown coefficients $\{c_0, c_1, \ldots, c_m\}$ can be determined in such a way that the following functional is minimized:

$$J(c_0, c_1, \ldots, c_m) = \sum_{i=1}^{N} \left[ f^{\text{exact}}(x_i, \eta, c_0, c_1, \ldots, c_m) - h(t) \right]^2.$$
Here, \( U(x_t, \eta, c_0, c_1, \ldots, c_m) \) are the calculated partial pressures. These quantities are determined from the solution of the direct problem given previously by using an approximated \( \hat{\alpha}(t) \) for the exact \( \alpha(t) \).

The estimated values of \( c_j, j = 1, \ldots, m \) are determined until the value of \( J(c_0, c_1, \ldots, c_m) \) is minimum. The computational procedure for estimating unknown coefficients \( c_j \) are described as follows:

Consider the following deterministic optimization problem

\[
\min_{C \in \mathbb{R}^n} J(C) = J(C^*) = J^*,
\]

where \( J(C) \) is real-valued bounded function defined on a closed bounded domain \( D \subset \mathbb{R}^n \) and \( C = (c_0, c_1, \ldots, c_m) \). It is assumed that \( J \) achieved its minimum value at a unique point \( C^* \). The function \( J(C) \) may have many local minimum in \( D \) but only one global minimum. When \( J(C) \) and \( D \) have some attractive properties, for instance, \( J(C) \) is a differentiable concave function and \( D \) is a convex region, then a local maximum is also a global maximum and problem (12) can be solved explicitly by mathematical programming methods (Rubinstein, R.Y., 1981). The formula (12) reveals that the paper, in fact, finds the best fit of the unknown \( \alpha(t) \) by the method of least squares. If the problem cannot be solved explicitly, then numerical methods based on random sampling, in particular Monte Carlo methods, can be applied (Farnoosh, R. and M. Ebrahimi, 2010).

**Numerical Experiments:**

In this section, we are going to demonstrate some numerical results for determining \((u, \alpha(t))\) in the inverse problem (1)-(5). All the computations are performed on the PC. However, to further demonstrating the accuracy and efficiency of this method, the present problem is investigated and an example is illustrated. Therefore the following example is considered and the solution is obtained.

**Example:**

In a clinical setting, we have the set of experimental data

\[
u_x = \alpha(t)u_x, \quad 0 < x < 1, \quad 0 < t < t_f,
\]

\[
u(x,0) = \sin(\pi x), \quad 0 < x < 1,
\]

\[
u(0,t) = 0, \quad 0 < t < t_f,
\]

\[
u(1,t) = 0, \quad 0 < t < t_f,
\]

\[
u(0.5,t) = h(t), \quad 0 < t < t_f,
\]

The diffusion equation (13) written for the partial pressure of gas process in the representative elementary volume (REV) when \( \alpha(t) \) is the time dependent effective diffusion coefficient of the alveolar region. Self-sufficiency of the REV allows us to impose the boundary condition of the REV being impermeable to the gas in question, that is, to require the mass flux through the REV boundary to be zero, \( u(0,t) = 0 \). Also, the red blood cells (RBC), distributed by random within the REV, are modelled as internal sinks of the gas, and consequently, can be viewed as another boundary condition of the gas partial pressure being always zero at the RBC's sites, \( u(1,t) = 0 \), [12]. The initial condition of the partial pressure is \( u(x,0) = \sin(\pi x) \).

The exact solution of the problem (13)-(17) is

\[
U(x,t) = \exp\left(-\pi^2 \int_0^t \alpha(y) \, dy\right) \sin(\pi x),
\]

and
\[
a(t) = -\frac{h'(t)}{\pi^2 h(t)}.
\]

While \( h(t) \) is positive, continuously differentiable, \( h'(t) \) is negative, and \( h(0) = 1 \), for example \( h(t) = \frac{1}{1 + t} \).

Figure 1 is presented to show the plot of error between \( \hat{a}(t) \) and \( a(t) \), when the polynomial with degree \( m = 3 \) is used to approximate \( a(t) \). Figure 2 is presented to show the error for the medium partial pressure \( u(x,t) \) between the exact solution of the problem (13)-(17) and numerical results that obtained by using the algorithm of the present work. We also employed the function estimator \( \hat{a}(t) = c_0 \cos(t) + c_2 \sin(t) \), in present algorithm and conclude the Figures 3,4, and 5. Figure 3 show the error for time dependent diffusion coefficient \( a(t) \) and Figure 4 show the error for the medium partial pressure \( u(x,t) \). Figure 5 is perform to show the plot of error between \( \hat{a}(t) \) and \( a(t) \), when \( t_f = 10 \).

![Fig. 1: Results for error between \( a(t) \) and \( \hat{a}(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 \) when \( t_f = 1 \).](image1)

![Fig. 2: Results for error between exact \( u(x,t) \) and present numerical results with \( t_f = 1 \) when polynomial form is used.](image2)
Fig. 3: Results for error between $a(t)$ and $\hat{a}(t) = c_1 \cos(t) + c_2 \sin t$ when $t_f = 1$.

Fig. 4: Results for error between exact $u(x,t)$ and present numerical results with $t_f = 1$ when trigonometric polynomial form is used.

Fig. 5: Results for error between $a(t)$ and $\hat{a}(t) = c_1 \cos(t) + c_2 \sin t$ when $t_f = 10$.
Conclusions and Future Directions:
1) The present study, successfully applies the stochastic method involving the Feynman–Kac formula to a one-dimensional parabolic inverse problem.
2) From the illustrated example it can be seen that the proposed stochastic method is efficient and accurate to estimate the unknown effective diffusivity in a one-dimensional parabolic inverse problem.
3) The results presented here suggest that the synthesis of the Feynman-Kac formula provides a promising probabilistic approach to parabolic inverse problem of the theory of gas transfer. The advantages of this approach include, but are not limited to, versatility, the possibility of computing the functions of interest at isolated points without computing them on massive meshes, and the opportunity of having simple scalable implementations with practically unlimited capability for parallel processing.
4) Here we explored the basic ideas of the Feynman-Kac formula in conjunction with Monte Carlo methods to a gas flow problem. In future papers we plan to extend the approach to nonlinear inverse problems and linear inverse problems with high dimensions.

REFERENCES