Eigenvalue Problems for the p-Laplace Operator with Nonlinear Boundary Condition

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Abstract: We study the following nonlinear eigenvalue problem with nonlinear boundary conditions

\[
\begin{align*}
-\Delta_p u &= \lambda a(x)|u|^{p-2}u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} |\nabla u|^{p-2} + \alpha |u|^{p-2}u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where \(\Omega\) is a bounded region in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), \(n\) denotes the unit outward normal vector on \(\partial \Omega\), \(a(x)\) is an indefinite weight function and \(\lambda\) is a real parameter. The properties of the first eigenvalue and the associated eigenfunction are examined. Our approach is motivated by (G. A. Afrouzi, 1988).

Key words: Eigenvalue; The p-Laplacian; Nonlinear Robin boundary condition.

INTRODUCTION

Eigenvalue problems for the p-Laplace operator subject to zero Dirichlet conditions on a bounded domain have been studied extensively during the past two decades and many interesting results have been obtained. The growing attention in the study of the p-Laplace operator is motivated by the fact that it arises in various applications, e.g., in the theory of non-Newtonian fluids both for the case \(p \geq 2\) (dilatant fluids) and the case \(1 < p < 2\) (pseudo-plastic fluids), reaction diffusion problems, flow through porous media, glacial sliding, theory of superconductors, biology, etc. (G. Astarita, 1974; F. Cirstea, 2001; R.E. Showalter, 1997).

Many results have been obtained on the structure of the spectrum of the boundary value problem. It is shown in (A. Le, 2006) using the Ljusternik-Schnirelman principle for the existence of a nondecreasing sequence of nonnegative eigenvalues for Dirichlet, Neuman, Robin and other boundary conditions. He proved the simplicity, isolation, boundedness and regularities of the principal eigenvalue. We remark that in the case \(N = 1\) fairly complete information on the spectra of the Dirichlet, Neuman and periodic boundary problem is available (P. Drábek, 1999; I. Nečas, 1971). In this paper we consider the nonlinear eigenvalue problem

\[
\begin{align*}
-\Delta_p u &= \lambda a(x)|u|^{p-2}u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} |\nabla u|^{p-2} + \alpha |u|^{p-2}u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

that is related to the class of nonlinear boundary problems arising in the theory of conformal transformation of Riemannian matrices. We shall investigate how the principal eigenvalues of (1) depend on \(\alpha\), obtaining new results for the cases \(\alpha = 0\), \(\alpha > 0\) and \(\alpha < 0\). The case \(\alpha < 0\) seems to have been considered far less often than the case \(\alpha \geq 0\), probably because it is more natural that the flux across the boundary should be outwards if there is a positive concentration at the boundary, and also because \(\alpha \geq 0\) is an easier condition to use when applying the maximum principle to discuss positive solutions. Our approach in this paper follows the technique of Hess and Kato (1980). However, the problem in this paper is nonlinear with nonlinear boundary condition and the associated operator is not selfadjoint.

In the next section we prove the existence and multiplicity of the principal eigenvalues of (1) depend on \(\alpha\), and then we present some remarks and open problems.

Main Results:

In this section we will setup an appropriate functional analysis framework for our problem. We work in the Sobolev space \(X = W^{1,p}(\Omega)\) with an ordinary norm

\[\|u\|_X = \left(\int_{\Omega} (|\nabla u|^p + |u|^p) \, dx\right)^{\frac{1}{p}}.\]

We consider for fixed \(\lambda\), the eigenvalue problem

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\[
\begin{align*}
-\Delta_p u - \lambda a(x)|u|^{p-2}u &= \mu |u|^{p-2}u \quad \text{in} \ \Omega \\
\frac{\partial u}{\partial \nu} |\nabla u|^{p-2} + a|u|^{p-2}u &= 0 \quad \text{on} \ \partial \Omega
\end{align*}
\] (2)

We denote the lowest eigenvalue of (2) by \(\mu(\alpha, \lambda)\). Let

\[A_{\alpha, \lambda}(\Phi) = \int_{\Omega} (|\nabla \Phi|^p - \lambda a(x)|\Phi|^p) \, dx + \alpha \int_{\partial \Omega} |\Phi|^p \, d\sigma.
\]

When \(\alpha > 0\), it is clear that \(A_{\alpha, \lambda}(\Phi)\) is bounded from below for all \(\Phi \in X; \ ||\Phi||_{L^p(\Omega)} = 1\). By using variational method in \([12]\), it can be shown that

\[\mu(\alpha, \lambda) = \inf_{\Phi \in X; ||\Phi||_{L^p(\Omega)}} A_{\alpha, \lambda}(\Phi)
\]

and eigenfunction corresponding to \(\mu(\alpha, \lambda)\) does not change sign on \(X\). Thus, \(\lambda\) is a principal eigenvalue of the problem (1) if and only if \(\mu(\alpha, \lambda) = 0\).

When \(\alpha < 0\), the boundedness from below of \(A_{\alpha, \lambda}(\Phi)\) is no longer obvious a priori, but is a consequence of the following lemma.

**Lemma 1:**

For every \(\varepsilon > 0\) there exists a constant \(C(\varepsilon) > 0\) such that

\[\int_{\partial \Omega} |\Phi|^p \, d\sigma \leq \varepsilon \int_{\Omega} |\nabla \Phi|^p \, dx + C(\varepsilon) \int_{\Omega} |\Phi|^p \, dx
\]

for all \(\Phi \in X\).

**Proof:**

Suppose that the result does not hold, i.e., there exist \(\varepsilon_0 > 0\) and a sequence \(\Phi_n \in X\) such that \(||\Phi_n||_{L^p(\Omega)} = 1\) and

\[\int_{\partial \Omega} |\Phi_n|^p \, d\sigma \geq \varepsilon_0 \int_{\Omega} |\nabla \Phi_n|^p \, dx + C(\varepsilon_0) \int_{\Omega} |\Phi_n|^p \, dx
\]

Suppose first that \(\{||\Phi_n||_{L^p(\Omega)}\}\) is unbounded. Then we assume without loss of generality that \(||\Phi_n||_{L^p(\Omega)} \to \infty\). Hence using Sobolev embedding theorem, we obtain \(||\Phi_n||_{L^p(\Omega)} \to \infty\), which is impossible.

Suppose now that \(\{||\Phi_n||_{L^p(\Omega)}\}\) is bounded. Then since \(\{\Phi_n\}\) is bounded in \(X\), we may assume without loss of generality that \(\Phi_n \to \Phi_0\) in \(X\) for some \(\Phi_0 \in X\). Since \(X\) may be compact embedded in \(L_p(\Omega)\) and \(L_p(\partial \Omega)\), it follows that \(\Phi_n \to \Phi_0\) in \(L_p(\Omega)\) and \(L_p(\partial \Omega)\). Thus \(\{\Phi_n\}\) is bounded in \(L_p(\partial \Omega)\). It follows from (3) that \(\Phi_n \to 0\) in \(L_p(\Omega)\) and so in \(L_p(\partial \Omega)\), but this is impossible because of (3).

**Corollary 2:**

If \(\alpha < 0\), then the functional \(A_{\alpha, \lambda}\) is bounded from below for any \(\Phi \in X; ||\Phi||_{L^p(\Omega)} = 1\). **Proof.** Choosing \(\varepsilon > 0\) such that \(\varepsilon < -\frac{1}{\alpha}\). By using Lemma 1, there exists \(C(\varepsilon) > 0\) such that

\[\int_{\partial \Omega} |\Phi|^p \, d\sigma \leq \varepsilon \int_{\Omega} |\nabla \Phi|^p \, dx + C(\varepsilon) \int_{\Omega} |\Phi|^p \, dx
\]

for all \(\Phi \in X\). Then

\[\int_{\Omega} (|\nabla \Phi|^p - \lambda a(x)|\Phi|^p) \, dx + \alpha \int_{\partial \Omega} |\Phi|^p \, d\sigma
\]

\[\geq \int_{\Omega} |\nabla \Phi|^p \, dx + \alpha \varepsilon \int_{\Omega} |\nabla \Phi|^p \, dx + C(\varepsilon) \int_{\Omega} |\Phi|^p \, dx - \lambda \int_{\Omega} a(x)|\Phi|^p \, dx
\]
\[
\geq a \epsilon \int_{\Omega} |\nabla \Phi|^p \, dx + a C(\epsilon) \int_{\Omega} |\Phi|^p \, dx - \lambda \int_{\Omega} \alpha(x)|\Phi|^p \, dx \\
\geq \min \{a \epsilon, a C(\epsilon)\} \left(\int_{\Omega} |\Phi|^p \, dx\right) - \lambda \int_{\Omega} \alpha(x)|\Phi|^p \, dx \\
\geq C'(\alpha, \epsilon, \lambda, \alpha) \left(\int_{\Omega} |\Phi|^p \, dx\right),
\]

where
\[
C'(\alpha, \epsilon, \lambda, \alpha) = \min \{a \epsilon, a C(\epsilon)\} - \lambda \sup_{x \in \Omega} |\alpha(x)|,
\]
For \(\lambda > 0\) and
\[
C'(\alpha, \epsilon, \lambda, \alpha) = \min \{a \epsilon, a C(\epsilon)\} - \lambda \inf_{x \in \Omega} |\alpha(x)|,
\]
For \(\lambda \leq 0\). This concludes the proof. 

Now since \(A_{\alpha, \lambda}(\Phi)\) is bounded from below, it follows that
\[
\mu(\alpha, \lambda) = \inf_{\Phi \in \mathcal{X}; \|\Phi\|_{L^p(\Omega)} = 1} A_{\alpha, \lambda}(\Phi)
\]
and that an eigenfunction corresponding to \(\mu(\alpha, \lambda)\) does not change sign on \(\Omega\). (A. Le, 2006; P. Lindqvist, 1990).

It is easy to see that \(\lambda \to \mu(\alpha, \lambda)\) is an affine and so a concave function.

**Lemma 3:**
\[
\mu(\alpha, \lambda) \to -\infty \text{ as } \lambda \to \pm \infty.
\]

**Proof:**
Consider \(u_1, u_2 \in \mathcal{X}\) with \(\|u_1\|_{L^p(\Omega)} = \|u_2\|_{L^p(\Omega)} = 1\), such that \(\int_{\Omega} \alpha(x)|u_1|^p \, dx > 0\) and \(\int_{\Omega} \alpha(x)|u_2|^p \, dx < 0\). It can be seen that \(\mu(\alpha, \lambda) \to -\infty\) as \(\lambda \to \pm \infty\).

Thus \(\lambda \to \mu(\alpha, \lambda)\) is an increasing function until it attains its maximum, and is a decreasing function thereafter.

As can be seen from variational characterization of \(\mu(\alpha, \lambda)\) that \(\mu(\alpha, 0) > 0\) for \(0 < \alpha < \infty\), and so \(\mu(\alpha, \lambda)\) has exactly two zeroes for \(0 < \alpha < \infty\). Thus in this case (1) has exactly two principal eigenvalues; one positive and one negative.

In the case \(\alpha \leq 0\), we have
\[
\mu(\alpha, 0) = \inf_{\Phi \in \mathcal{X}; \|\Phi\|_{L^p(\Omega)} = 1} A_{\alpha, 0}(\Phi)
\]
\[
\leq \inf_{\Phi \in \mathcal{X}; \|\Phi\|_{L^p(\Omega)} = 1} A_{0,0}(\Phi)
\]

Since for the constant function \(\Phi = \frac{1}{|\Omega|^\frac{1}{p}}\), we have the following:
\[
\Phi \in \mathcal{X}; \|\Phi\|_{L^p(\Omega)} = 1 \text{ and } A_{0,0}(\Phi) = 0,
\]
thus we deduce that \(\inf_{\Phi \in \mathcal{X}; \|\Phi\|_{L^p(\Omega)} = 1} A_{0,0}(\Phi) \leq 0\) and so \(\mu(\alpha, 0) \leq 0\).

**Lemma 4:**
Suppose that \(u_0\) is a positive eigenfunction of (2) corresponding to the principal eigenvalue \(\mu(\alpha, \lambda)\). Then
\[
\frac{d\mu}{d\lambda}(\alpha, \lambda) = \frac{\int_{\Omega} \alpha(x)|u_0|^p \, dx}{\int_{\Omega} |u_0|^p \, dx}.
\]
provided that
\[ u_0 \nabla u_0 \nabla u_0 = |\nabla u_0|^2 u_0', \tag{4} \]
where \( u_0' = \frac{du_0}{dx}(x, \lambda). \)

**Proof:**

First note that by simplicity of the principal eigenvalue \( \mu(\alpha, \lambda) \) \([7]\), \( u_0' = \frac{du_0}{dx}(x, \lambda) \) there exists. Regarding \( u_0 \) and \( \mu \) as functions of \( \lambda \), we have

\[
\begin{aligned}
-\Delta u_0 - \lambda a(x) u_0^{p-1} & = \mu u_0^{p-1} \quad \text{in } \Omega \\
\frac{\partial u_0}{\partial n} |\nabla u_0|^{p-2} + \alpha u_0^{p-1} & = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\tag{5}
\]

Then \( u_0' \) satisfies

\[
(p - 1) \left[ -\nabla (|\nabla u_0|^{p-2} \nabla u_0) - \lambda a(x) u_0^{p-2} - \mu u_0^{p-2} \right] = a(x) u_0^{p-1} \frac{du_0}{dx} u_0^{p-1},
\tag{6}
\]
in \( \Omega \), and

\[
(p - 2)|\nabla u_0|^2 \nabla u_0 \nabla u_0' \frac{\partial u_0}{\partial n} + |\nabla u_0|^2 \frac{\partial u_0'}{\partial n} + (p - 1) \alpha u_0^{p-2} u_0' = 0,
\tag{7}
\]
on \( \partial \Omega \).

Multiplying (6) by \( u_0 \) and integrating over \( \Omega \) gives

\[
(p - 1) \left[ (p - 2) \int_{\partial \Omega} |\nabla u_0|^2 \nabla u_0 \nabla u_0' \frac{\partial u_0}{\partial n} d\sigma + (p - 1) \alpha \int_{\partial \Omega} u_0^{p-1} u_0' d\sigma + ight. \\
\left. \int_{\Omega} |\nabla u_0|^2 \nabla u_0 \nabla u_0' dx - \lambda \int_{\Omega} a(x) u_0^{p-1} u_0' dx - \mu \int_{\Omega} u_0^{p-1} u_0' dx \right] = \int_{\Omega} a(x) u_0^p dx + \frac{du_0}{dx} \int_{\Omega} u_0^p dx.
\tag{8}
\]

Here the boundary condition in (7) is used.

We now claim that the left hand side of (8) vanishes which completes the proof.

Multiplying (5) by \( u_0 \) and integrating over \( \Omega \) gives

\[
\int_{\Omega} |\nabla u_0|^2 \nabla u_0 \nabla u_0' dx + \alpha \int_{\partial \Omega} u_0^{p-1} u_0' d\sigma - \lambda \int_{\Omega} a(x) u_0^{p-1} u_0' dx = \mu \int_{\Omega} u_0^{p-1} u_0' dx.
\]

So

\[
\int_{\Omega} |\nabla u_0|^2 \nabla u_0 \nabla u_0' dx - \lambda \int_{\Omega} a(x) u_0^{p-1} u_0' dx - \mu \int_{\Omega} u_0^{p-1} u_0' dx = -\alpha \int_{\partial \Omega} u_0^{p-1} u_0' d\sigma.
\tag{9}
\]

Substituting (9) in (8), gives

\[
(p - 1) \left[ (p - 2) \int_{\partial \Omega} |\nabla u_0|^2 \nabla u_0 \nabla u_0' \frac{\partial u_0}{\partial n} d\sigma + (p - 1) \alpha \int_{\partial \Omega} u_0^{p-1} u_0' d\sigma - \int_{\Omega} a(x) u_0^p dx + \frac{du_0}{dx} \int_{\Omega} u_0^p dx \right] = \\
\int_{\Omega} a(x) u_0^p dx - \alpha \int_{\partial \Omega} u_0^{p-1} u_0' d\sigma.
\tag{10}
\]

By direct calculation we have

\[
(p - 1) \left[ (p - 2) \alpha \int_{\partial \Omega} u_0^p \left( \frac{\nabla u_0 \nabla u_0'}{|\nabla u_0|^2} \right) d\sigma + (p - 2) \alpha \int_{\Omega} u_0^{p-1} u_0' d\sigma \right] = (p - 1)(p - 2) \alpha \int_{\partial \Omega} u_0^p \left( \frac{\nabla u_0 \nabla u_0'}{|\nabla u_0|^2} \right) d\sigma = 0.
\tag{11}
\]

Hence the left hand side of (10) vanishes and so the result follows. ■
The above lemma shows that where \( \lambda \to \mu(\alpha, \lambda) \) is an increasing (decreasing) function we have that \( \int_{\Omega} a(x) u_0^p \, dx < 0 (> 0) \), and at critical point we must have \( \int_{\Omega} a(x) u_0^p \, dx = 0 \), whenever \( u_0 \) satisfies the condition (4).

The next lemma shows that \( \lambda \to \mu(\alpha, \lambda) \) has a unique critical point.

**Lemma 5:**

Suppose that \( u_0 \) is an eigenfunction of (2) corresponding to the principal eigenvalue \( \mu(\alpha, \lambda_0) \), such that \( \int_{\Omega} a(x) u_0^p \, dx = 0 \). Then \( \mu(\alpha, \lambda_0) > \mu(\alpha, \lambda) \) for \( \lambda \neq \lambda_0 \).

**Proof:**

Regarding \( \mu \) as function of \( \alpha \) and \( \lambda \), we have

\[
\begin{aligned}
-\Delta u_0 - \lambda a(x) u_0^{p-1} &= \mu(\alpha, \lambda_0) u_0^{p-1} \quad \text{in } \Omega \\
\frac{\partial u_0}{\partial n} |\nabla u_0|^{p-2} + \alpha u_0^{p-1} &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(12)

multiplying (11) by \( u_0 \) and integrating over \( \Omega \) gives

\[
\int_{\Omega} |\nabla u_0|^p dx + \alpha \int_{\partial \Omega} u_0^p \, ds = \mu(\alpha, \lambda_0) \int_{\Omega} u_0^p \, dx.
\]

Let \( v_0 = \frac{u_0}{||u_0||_x} \). Then

\[
\mu(\alpha, \lambda_0) = \int_{\Omega} |\nabla v_0|^p dx + \alpha \int_{\partial \Omega} v_0^p \, ds,
\]

and

\[
\mu(\alpha, \lambda) \leq \int_{\Omega} |\nabla v_0|^p dx + \alpha \int_{\partial \Omega} v_0^p \, ds - \lambda \int_{\Omega} a(x)v_0^p \, dx
\]

\[
= \int_{\Omega} |\nabla v_0|^p dx + \alpha \int_{\partial \Omega} v_0^p \, ds = \mu(\alpha, \lambda_0).
\]

We now show that \( \mu(\alpha, \lambda) < \mu(\alpha, \lambda_0) \), whenever \( \lambda \neq \lambda_0 \).

Suppose otherwise. Then \( \mu = \mu(\alpha, \lambda) = \mu(\alpha, \lambda_0) \) satisfies

\[
-\Delta u_0 - \lambda_0 a(x) u_0^{p-1} = \mu u_0^{p-1} \quad \text{in } \Omega; \quad \frac{\partial u_0}{\partial n} |\nabla u_0|^{p-2} + \alpha u_0^{p-1} = 0 \quad \text{on } \partial \Omega,
\]

and

\[
-\Delta u_0 - \lambda a(x) u_0^{p-1} = \mu u_0^{p-1} \quad \text{in } \Omega; \quad \frac{\partial u_0}{\partial n} |\nabla u_0|^{p-2} + \alpha u_0^{p-1} = 0 \quad \text{on } \partial \Omega,
\]

while, \( \lambda \neq \lambda_0 \), and this is a contradiction. \( \blacksquare \)

The above result shows that the unique global maximum of \( \lambda \to \mu(\alpha, \lambda) \) occurs when \( \lambda = \lambda_0 \).

Hence the graph of \( \lambda \to \mu(\alpha, \lambda) \) may have 2, 1 and 0 intersections with the \( \lambda \)-axis, and so (1) may have 2, 1 and 0 principal eigenvalues.

As we shall see in the next theorem when \( \alpha > 0 \), (1) has 2 principal eigenvalues, one positive and one negative.

When \( \alpha = 0 \), i.e., we have nonlinear Neumann boundary condition, \( \mu(0, 0) = 0 \) and the corresponding eigenfunction is constant. Hence \( \frac{d\mu}{d\alpha}(0) > 0 (= 0)(< 0) \) as \( \int_{\Omega} a(x) \, dx < 0 (= 0)(> 0) \). Thus when \( \alpha = 0 \), \( \mu = 0 \) is a principal eigenvalue in all cases; if \( \int_{\Omega} a(x) \, dx < 0 \), there is an additional positive principal eigenvalue; and, if \( \int_{\Omega} a(x) \, dx > 0 \), there is an additional negative principal eigenvalue and, if \( \int_{\Omega} a(x) \, dx = 0 \), \( \mu = 0 \) is the only principal eigenvalue. We now consider what happens when \( \alpha < 0 \). We first assume that \( \int_{\Omega} a(x) \, dx < 0 \). It is clear from a variational characterization of \( \mu(\alpha, \lambda) \) that \( \alpha \to \mu(\alpha, \lambda) \) is a strictly increasing, concave function. Thus for \( \alpha \) sufficiently small and negative (1) has two positive principal eigenvalues.
Lemma 6:

There exists $\alpha^* < 0$ such that (1) has no principal eigenvalues if $\alpha < \alpha^*$.

Proof:

Suppose $\alpha < 0$ and $u_0$ is a positive eigenvalue of (1) corresponding to a positive principal eigenvalue $\lambda_0$. Note that because of boundedness and regularity of eigenfunction $u_0$ (A. Le, 2006), we can use $u_0^{1-p}$ as a test function. Using the maximum principle, it follows that $u_0(x) > 0$ for all $x \in \Omega$. Since $\alpha < 0$, we have $\mu(\alpha, \lambda_0) < \mu(0, \lambda_0)$, i.e., $\mu(\alpha, \lambda_0) > 0$.

Hence $\lambda_0 < \mu_0$ (the positive eigenvalue of the Neumann problem). Dividing (1) by $u_0^{p-1}$ and integrating over $\Omega$, we have

$$
\int_{\Omega} \frac{-\Delta u_0}{u_0^{p-1}} \, dx = \lambda_0 \int_{\Omega} a(x) \, dx
$$

By direct calculation

$$
\int_{\Omega} \frac{-\Delta u_0}{u_0^{p-1}} \, dx = \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \, dx - \int_{\partial \Omega} \nabla u_0 |^{p-2} \frac{\partial u_0}{\partial n} u_0^{1-p} \, d\sigma = (1 - p) \int_{\Omega} \frac{|\nabla u_0|^p}{u_0} \, dx + \alpha |\partial \Omega|.
$$

Hence

$$
\alpha = (p - 1) \int_{\Omega} \frac{|\nabla u_0|^p}{u_0} \, dx + \lambda_0 \int_{\Omega} a(x) \, dx / |\partial \Omega|.
$$

Since $\lambda_0 < \mu_0$, $\alpha$ cannot be too negative, and the proof is complete. \hfill \blacksquare

It follows that for large negative $\alpha$ the graph of $\lambda \to \mu(\alpha, \lambda)$ does not touch the $\lambda$-axis, and so by the continuity of $\alpha \to \mu(\alpha, \lambda)$ (because of concavity), there must exist $\alpha_0$ such that $\max_{\lambda} \mu(\alpha_0, \lambda) = 0$. Clearly the problem (1) at $\alpha_0$ has precisely one principal eigenvalue.

A similar analysis can be carried out in the case $\mu(\alpha, \lambda)$; in this case two negative principal eigenvalues will occur for an appropriate range of negative $\alpha$.

Our results may be summarized in the following theorem.

Theorem 1:

There exists $\alpha_0 \leq 0$ such that
(i) if $\alpha < \alpha_0$, then (1) does not have a principal eigenvalue;
(ii) $\alpha = \alpha_0$, then (1) has a unique principal eigenvalue with corresponding eigenfunction $u_0$;
(iii) if $\alpha > \alpha_0$, then (1) has exactly two principal eigenvalues;
(iv) $\alpha_0 = 0$ if and only if $\int_{\Omega} a(x) \, dx = 0$.

The following theorem gives us another property of the eigenfunctions.

Theorem 2:

Let $\alpha \geq 0$ and suppose that $\lambda_0 \neq 0$ is a principal eigenvalue of (1) with corresponding positive principal eigenfunction $u_0$. Then $\lambda_0 \int_{\Omega} a(x) u_0^p \, dx > 0$ for all $v \geq 1$.

Proof:

Suppose $\alpha > 0$. Multiplying (1) by $u_0^\gamma$ where $\gamma \geq 1$, we obtain

$$
(-\Delta u_0)u_0^\gamma = \lambda_0 a(x) u_0^{p+\gamma-1}
$$

on $\Omega$ and so

$$
\gamma \int_{\Omega} |\nabla u_0|^p u_0^{\gamma-1} \, dx - \int_{\partial \Omega} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} u_0^{\gamma} \, d\sigma = \lambda_0 \int_{\Omega} a(x) u_0^{p+\gamma-1} \, dx.
$$

Hence

$$
\lambda_0 \int_{\Omega} a(x) u_0^{p+\gamma-1} \, dx = \int_{\Omega} |\nabla u_0|^p u_0^{\gamma-1} \, dx + \alpha \int_{\partial \Omega} u_0^{p+\gamma-1} \, d\sigma
$$

and so the required result holds.
If $\alpha = 0$ the surface integral term vanishes and the result follows easily.

**Remark 1:**
For the Dirichlet problem, as mentioned by Lindqvist (1990) and Anane and Tsouli (1994), we do not need any regularity of $\partial \Omega$. The Robin boundary condition, which includes the Neumann problem (when $\alpha = 0$) requires $\partial \Omega$ be of class $C^1$. Moreover in the case $\alpha \neq 0$, it requires $\partial \Omega$ be of class $C^1$. In order to have the simplicity and isolation of the principal eigenvalue.

**Remark 2:**
An Le in (2006) proved the existence of L-S eigenvalues. There are still some interesting problems that we have not answered, that: Are the L-S eigenvalues different from the eigenvalues that we find in this paper?

**Remark 3:**
Lieberman (1988) shows that on the special assumption on $g$ and $\psi$, the solution of the problem

$$
\begin{align*}
-\Delta_p u &= g(x) & \text{in } \Omega \\
\frac{\partial u}{\partial n} &+ |u|^{p-2} \psi(x, u) = 0, & \text{on } \partial \Omega
\end{align*}
$$

is bounded. Also we have not answered the interesting problem that What type of conditions on $\psi(x, u)$ leads to the existence of the eigenvalues of the problem

$$
\begin{align*}
-\Delta_p u &= \lambda a(x)|u|^{p-2}u & \text{in } \Omega \\
\frac{\partial u}{\partial n} &+ |u|^{p-2} + \alpha |u|^{p-2}u = 0 & \text{on } \partial \Omega
\end{align*}
$$

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