An Approximation Method Based on Matrix Formulated Algorithm for the Numerical Study of a Biharmonic Equation

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Abstract: In this research, a New matrix formulation technique based on the arbitrary polynomial base is proposed for solving the Biharmonic equation with two initial and two boundary conditions. We introduce An original method for the numerical solution of Biharmonic equation with constant coefficients. The method is based on finding a solution in the form of a polynomial in two variables \( U_{ij}(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} U_{ij} x^i t^j \) with undetermined coefficients \( U_{ij} \). By using the operational matrix of derivative, we reduce the problem to a set of linear algebraic equations. We utilize an interesting numerical process for numerical section of the method. Finally, the performance of the proposed method is investigated by considering several biharmonic problems.

Key words: Matrix formulation method; Hyperbolic equation; Biharmonic equation; Expansion methods.

INTRODUCTION

The partial differential equations with supplementary conditions is one of the most important branches of the applied sciences and many authors paid much attention to solving this case of equations.

In this paper a new matrix formulation based on the operational matrix defined in (Soltanalizadeh, B., 2011) is presented for the Biharmonic equation,

\[
\frac{\partial^4 u}{\partial t^4} + a \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^4 u}{\partial x^2} = f(x,t), \quad (x, y) \in (a,b) \times (a,b),
\]

(1)

With the boundary conditions,

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(x,a)}{\partial t} = r(x), \\
\frac{\partial u(x,b)}{\partial t} = r'(x)
\end{array} \right. \\
(2)
\end{align*}
\]

And

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(x,a)}{\partial x} = s(x), \\
\frac{\partial u(x,b)}{\partial x} = s'(x)
\end{array} \right. \\
(3)
\end{align*}
\]

And

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(a,t)}{\partial t} = p(t), \\
\frac{\partial u(b,t)}{\partial t} = p'(t)
\end{array} \right. \\
(4)
\end{align*}
\]

And

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(a,t)}{\partial x} = q(t), \\
\frac{\partial u(b,t)}{\partial x} = q'(t)
\end{array} \right. \\
(5)
\end{align*}
\]

where the functions \( f(x,t), \quad r(x), \quad r'(x), \quad s(x), \quad s'(x), \quad p(t), \quad p'(t), \quad q(t) \) and \( q'(t) \) and the constants \( \alpha, \beta, a \) and \( b \) are known.
In 1951, Timoshenko and Goodier (Timoshenko et al., 1951), presented the Biharmonic equation. Recently, Mai-Duy (Duy et al., 2005) reported a new indirect radial-basis-function collocation method for solving numerically the biharmonic boundary value problems. Dehghan and Mohebbi (Dehghan et al., 2006) proposed two compact finite difference approximations algorithms for solving numerically the three-dimensional biharmonic equation with Dirichlet boundary conditions. Mai-Duy and Tanner (Duy et al., 2007) suggested a new spectral collocation method for solving numerically the two-dimensional biharmonic boundary value problems. Authors of (Ali et al., 2007) utilized Variational iteration method to find the solutions of two-dimensional biharmonic equations. A new boundary-integral-equation method for numerically solving biharmonic problems with Dirichlet boundary conditions has been presented in (Duy et al., 2006).

A new matrix formulation technique with arbitrary polynomial bases has been proposed for the numerical/analytical solution of the Heat equations with nonlocal boundary conditions (Soltanalizadeh, 2011) and Two matrix formulation techniques based on the shifted standard and shifted Chebyshev bases are proposed for the numerical solution of the wave equation with the non-local boundary condition (Ghehsareh et al., 2011). There are some similar methods, such as differential transformation method (Soltanalizadeh, 2011, Soltanalizadeh et al., 2011, 2012, Karimi et al., 2011, Abazari et al., 2010, Abazari et al., 2011, Dogan et al., 2011), Tau method (Tari et al., 2008, Karimi et al., 2011) and other methods (Abbasbandy et al., 2009, Khan et al., 2011, Saadatmandi et al., 2007, Yildirim, 2010).

This paper has the following structure: The properties of matrix formulation method by convert the problem to the some linear algebraic equations have been presented in section 2. The numerical process has been applied in section 3. by selecting some numerical tests.

**Matrix Formulation:**

In (1-5), the functions \( f(x,t), r(x), r'(x), s(x), s'(x), p(t), p'(t), q(t) \) and \( q'(t) \) generally are not polynomials. We assume that these functions are polynomial or they can be approximated by polynomials to any degree of accuracy. Then by using Taylor series, we can write:

\[
\begin{align*}
f(x,t) &\approx \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y^j t^i = X^T FT, \\
r(x) &\approx \sum_{j=0}^{\infty} x^j = X^T R, r'(x) \approx \sum_{j=0}^{\infty} x^j = X^T R', \\
s(x) &\approx \sum_{j=0}^{\infty} s^j x = X^T S, s'(x) \approx \sum_{j=0}^{\infty} s^j x = X^T S', \\
p(t) &\approx \sum_{j=0}^{\infty} p^j t = PT, p'(t) \approx \sum_{j=0}^{\infty} p^j t = PT', \\
q(t) &\approx \sum_{j=0}^{\infty} q^j t = QT, q'(t) \approx \sum_{j=0}^{\infty} q^j t = QT', \\
\end{align*}
\]

where

\[
X = [1, x, x^2, ..., x^n], T = [1, t, t^2, ..., t^n]^T, R = [r_0, r_1, r_2, ..., r_n]^T, \\
R' = [r_0', r_1', r_2', ..., r_n']^T, S = [s_0, s_1, s_2, ..., s_n]^T, S' = [s_0', s_1', s_2', ..., s_n']^T, \\
P = [p_0, p_1, p_2, ..., p_n]^T, P' = [p_0', p_1', p_2', ..., p_n']^T, \\
Q = [q_0, q_1, q_2, ..., q_n]^T, Q' = [q_0', q_1', q_2', ..., q_n']^T, \\
F = [F_0, F_1, F_2, ..., F_n], F_i = [f_{i0}, f_{i1}, f_{i2}, ..., f_{in}]^T, \quad i = 0, 1, 2, ..., n.
\]

Therefore we consider approximate solution of the form,

\[
U(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} x^i t^j = X^T UT,
\]
where $U = [U_0, U_1, U_2, \ldots, U_n]$, with $U_i = [u_{i0}, u_{i1}, u_{i2}, \ldots, u_{in}]^T$. The matrix $U$ is an $(n+1) \times (n+1)$ matrix which contains $(n+1)^2$ unknown coefficients of $U_i(x,t)$.

Now, we recall the following lemma. This lemma is proved by induction.

**Lemma 2.1. The effect of Repeated Differentiation on Coefficients Vector:**

Let $a = [a_0, a_1, a_2, \ldots, a_n]$ of a polynomial $y_a(x) = aX$ be the same as that of post-multiplication of $a$ by the matrix $\eta'$:

$$\frac{d'}{dx'} y_a(x) = a\eta' X,$$

where $\eta$ is the $(n+1) \times (n+1)$ operational matrix of derivation:

$$\eta = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 2 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 3 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & n & 0 \\
\end{bmatrix}_{(n+1)(n+1)}$$

**Proof:** See (Ortiz et al., 1981).

We first consider Eqs. (2). By using Eqs. (6) and (7) in (2), we have

$$X^T UT (a) = X^T R, X^T UT (b) = X^T R',$$

which implies

$$UT (a) = R, \quad (8)$$

$$UT (b) = R'. \quad (9)$$

since $T$ is a basis vector.

Applying Lemma 2.1 and Eq. (6) and (7) in (3), we get

$$X^T \eta UT (a) = X^T S, X^T \eta UT (b) = X^T S',$$

then we have

$$\eta UT (a) = S, \quad (10)$$

$$\eta UT (b) = S'. \quad (11)$$

By applying a similar process on the Eq. (4), we have

$$X^T (a) UT = PT, X^T (b) UT = PT'$$

which implies

$$X^T (a) U = P, \quad (12)$$
And by using Lemma 2.1 and Eq. (6) and (7) in (5), we get
\[ X^T (a) U \eta^T T = QT, X^T (b) U \eta^T T = Q'T \]

thus
\[ X^T (a) U \eta^T = Q, \]

\[ X^T (b) U \eta^T = Q'. \]  

Finally, we consider Eq. (1), by corollary 2.3. and Eq. (4), we obtain
\[ X^T U \eta^T T + \alpha X^T (\eta^T)^{\gamma} U \eta^T T + \beta X^T (\eta^T)^{\gamma} U T = X^T F T, \]

hence
\[ U \eta^\gamma + \alpha (\eta^\gamma)^{\gamma} U \eta^\gamma + \beta (\eta^\gamma)^{\gamma} U = F \]  

In this section, we arrange the linear equations obtained by Eqs. (8)-(16) to have a system of \((n + 1)^2\) equations for the \((n + 1)^2\) unknowns. We find \((n + 1)\) equations from (8) and \(n + 1\) equations from Eq. (9).

Note that, since the equation (1) and (3)-(5) are not defined for \(x \in [a,b]\) and \(x \in [a,b]\), we chose \(n - 1\) equations from (10) and \(n - 1\) equations from (11) and \((n - 1)\) equation from (12) and \(n - 1\) equations from (13). Then \(n - 3\) equations from (14) and \(n - 1\) equations from (15) can be given. Finally, the remainder \((n - 3)^2\) equations must arranged from Eq. (16).

Now we have \((n + 1)^2\) linear algebraic equations that by solving this system, we, can find the values of \(u_{i,j}\) for \(i, j = 0, 1, \ldots, n\) that it implies the approximate of \(U(x, t)\) by using Eq. (7).

**Numerical Example:**

In this section we give some numerical examples to compute an approximate solution of Eqs. (1)-(5) by the methods discussed in this paper. Firstly, we define some of the errors as follows:

\[ \|u_{n,m} - u^*\|_{l,\infty} = \max \{\|u_{n,m}(x,t) - u^*(x,t)\|, \ 0 \leq t \leq T\}, \]

where \(u_n(x, t)\) is the computed result with \(n\).

**Example 1:** Consider the Eqs. (1)-(5)

\[
\begin{align*}
 f(x,t) &= -24(1 + x^3 - 5x^5) - 4(6x - 100x^6)(4 - 12r^2) - 1800x(2r^2 - t^2), \\
 r(x) &= r'(x) = (1 + x^3 - 5x^5), \\
 s(x) &= s'(x) = 0, \\
 p(t) &= 10r^2 - 5r^4, \\
 p'(t) &= -6t^2 + 3t^4, \\
 q(t) &= q'(t) = -44t^2 + 22r^4, \\
 a &= -4, b = 3, \alpha = -1, b = 1
\end{align*}
\]

By choosing \(n = 5\), we obtain

\[ u_5(x, t) = (1 + x^3 - 5x^5)(2r^2 - t^2), \]
which is the exact solution of the problem.

**Example 2:** Consider the following biharmonic equation

\[
\begin{aligned}
&f(x,t) = 0, \\
r(x) = \exp(x - 1), \\
r'(x) = \exp(x + 1), \\
s(x) = \exp(x - 1), \\
s'(x) = \exp(x + 1), \\
p(t) = \exp(t - 1), \\
p'(t) = \exp(t + 1), \\
q(t) = \exp(t - 1), \\
q'(t) = \exp(t + 1), \\
\alpha = -2, \beta = 1, a = -1, b = 1
\end{aligned}
\]

The exact solution of this problem is

\[
u(x,t) = \exp(x + t).
\]

**Table 1:** The maximum errors \(\| u_{n,m} - u^* \|_{l,\infty} \) from Example 2.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N = 20 )</th>
<th>( n = 30 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>1.129 \times 10^{-18}</td>
<td>3.206 \times 10^{-34}</td>
</tr>
<tr>
<td>-0.8</td>
<td>2.393 \times 10^{-19}</td>
<td>8.355 \times 10^{-34}</td>
</tr>
<tr>
<td>-0.6</td>
<td>6.347 \times 10^{-20}</td>
<td>1.188 \times 10^{-33}</td>
</tr>
<tr>
<td>-0.4</td>
<td>4.11 \times 10^{-20}</td>
<td>1.145 \times 10^{-33}</td>
</tr>
<tr>
<td>-0.2</td>
<td>4.723 \times 10^{-20}</td>
<td>7.476 \times 10^{-34}</td>
</tr>
<tr>
<td>0.0</td>
<td>4.637 \times 10^{-20}</td>
<td>1.006 \times 10^{-37}</td>
</tr>
<tr>
<td>0.2</td>
<td>5.304 \times 10^{-20}</td>
<td>1.787 \times 10^{-34}</td>
</tr>
<tr>
<td>0.4</td>
<td>5.881 \times 10^{-20}</td>
<td>2.264 \times 10^{-37}</td>
</tr>
<tr>
<td>0.6</td>
<td>7.675 \times 10^{-20}</td>
<td>2.026 \times 10^{-37}</td>
</tr>
<tr>
<td>0.8</td>
<td>1.059 \times 10^{-19}</td>
<td>1.164 \times 10^{-34}</td>
</tr>
<tr>
<td>1.0</td>
<td>1.950 \times 10^{-19}</td>
<td>3.412 \times 10^{-34}</td>
</tr>
</tbody>
</table>

**Conclusions:**

In this research we presented a numerical scheme for solving the Biharmonic equation with given initial and boundary conditions. By using the operational matrix of derivative, we transform the model of partial differential equation (PDE) into a system of first order, linear, ordinary differential equations (ODEs). The results obtained by using our scheme are very highly accurate when compared with those results which have already existed in the literature. As another advantage of this method, in spite of other similar methods such as finite difference, it is possible to compute the solution at each mesh point after finding the approximate solution.

**REFERENCES**


