Stability of Smart Beams with Varying Properties Based on the First Order Shear Deformation Theory Located on a Continuous Elastic Foundation

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Abstract: This paper studies stability of functionally graded beams with piezoelectric layers subjected to axial compressive load that is simply supported at both ends lies on a continuous elastic foundation. The displacement field of beam is assumed based on first order shear deformation beam theory. Applying the Hamilton’s principle, the governing equation is established. The influences of applied voltage, dimensionless geometrical parameter, functionally graded index, foundation coefficient and piezoelectric thickness on the critical buckling load of beam are presented. To investigate the accuracy of the present analysis, a compression study is carried out with a known data.

Key words: Mechanical Buckling, Functionally graded beam- Piezoelectric layer.

INTRODUCTION

Structural stability is considered to be one of the most important engineering issues in the design and application of slender structures. Buckling and postbuckling are the two main types of structural instability, they often govern the failure of structures under static or dynamic compressive loading conditions, thus, have been investigated by several researchers in the past decades (Ari-Gur, J., 1982). Piezoelectric materials have been used in the past few years in a variety of applications ranging from active control to noise suppression. In all these applications, piezoelectric actuators are used to enhance the performance of a structural system by inducing a favorable structural deformation. Detailed models on the iteration between piezoelectric sensors or actuators with the structure to which they are bonded or embedded have been developed (Tzou, H.S., G.C. Wan, 1990).

To the author’s knowledge, there is no analytical solution available in the open literatures for stability of functionally graded beams with piezoelectric layers located on a continuous elastic foundation based on first order shear deformation beam theory. In the present work, the stability of a functionally graded beam with piezoelectric actuators subjected to axial compressive loads located on a continuous elastic foundation based on first order shear deformation beam theory is studied. Applying the Hamilton’s principle, the equilibrium equations of beam are derived and solved. The effects of dimensionless geometrical parameter, foundation coefficient and functionally graded index on the critical buckling load of beam are presented. To investigate the accuracy of the present analysis, a compression study is carried out with a known data.

Formulation

The formulation that is presented here is based on first order shear deformation beam theory. Based on this theory, the displacement field can be written as (Wang C.M., J.N. Reddy, 2000):

\[ u(x, z) = z\phi(x) \quad , \quad w(x, z) = w_0(x, z) \]

In view of the displacement field, the strain displacement relations are given by (Wang C.M., J.N. Reddy, 2000):

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} = z\frac{d\phi}{dx} \quad , \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi + \frac{dw}{dx} \]

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Consider a functionally graded beam with piezoelectric actuators and rectangular cross-section as shown in Fig. 1. The thickness, length, and width of the beam are denoted, respectively, by \( h \), \( L \) and \( b \). Also, \( h_T \) and \( h_B \) are the thickness of top and bottom of piezoelectric actuators, respectively. The \( x-y \) plane coincides with the midplane of the beam and the \( z \)-axis located along the thickness direction.

The Young's modulus \( E \) is assumed to vary as a power form of the thickness coordinate variable \( z \) \((-h/2 \leq z \leq h/2)\) as follow (Karami, 2008):

\[
E(z) = (E_c - E_m) V + E_m, \quad V = \left( \frac{2z + h}{2h} \right)^k
\]

where \( k \) is the power law index and the subscripts \( m \) and \( c \) refer to the metal and ceramic constituents, respectively. The constitutive relations for functionally graded beam with piezoelectric layers based on first order shear deformation beam theory are given by (Wang C.M., J.N. Reddy, 2000):

\[
\sigma_{xx} = Q_{11}(z) e_{xx} - e_{31} E_x, \quad \sigma_{zz} = Q_{55}(z) e_{zz} - e_{15} E_z
\]

where \( E_i = \frac{V}{h_i} \) and \( \sigma_{xx}, \sigma_{zz}, Q_{11}(z), Q_{55}(z) \) are the normal, shear stresses and plane stress-reduced stiffnesses and \( e_{31}, e_{15} \) are piezoelectric elastic stiffnesses respectively. Also, \( u \) and \( w \) are the displacement components in the \( x \)- and \( z \)-directions, respectively.

The potential energy can be expressed as (Wang C.M., J.N. Reddy, 2000):

\[
U = \frac{1}{2} \int (\sigma_{xx} e_{xx} + \sigma_{zz} e_{zz}) \, dv
\]

Substituting Eq. (2) and Eq. (4) into Eq. (5) and neglecting the higher-order terms, we obtain

\[
U = \frac{b}{2} \int \left[ D \left( \frac{d\phi}{dx} \right)^2 + A \left( \frac{d\phi}{dx} + \frac{dw}{dx} \right)^2 + 2 \phi \frac{d\phi}{dx} - e_{31} (h_T V_T + h_B V_B) \frac{d\phi}{dx} - e_{15} (V_T + V_B) (\phi + \frac{dw}{dx}) \right] \, dx
\]

where \( A = \int_{-h/2}^{h/2} Q_{11}(z) \, dz \), \( D = \int_{-h/2}^{h/2} z^2 Q_{55}(z) \, dz \), \( V_T \) and \( V_B \) are the applied voltages on the top and bottom actuators respectively. The beam is subjected to the axial compressive loads. The work done by the axial compressive load can be expressed as (Wang C.M., J.N. Reddy, 2000):

\[
W = \frac{1}{2} \int P \left( \frac{d\phi}{dx} \right)^2 \, dx
\]

We apply the Hamilton's principle to derive the equilibrium equations of beam, that is (Wang C.M., J.N. Reddy, 2000):

\[
\frac{d}{dx} \left[ \frac{E(x) b}{2} \frac{d^2 u}{dx^2} \right] + F(x) = 0
\]
Substitution from Eqs. (6) and (7) into Eq. (8) leads to the following equilibrium equations of the functionally graded Engesser-Timoshenko beam with piezoelectric layers. Assume that a FG beam with piezoelectric actuators that is simply supported at both ends lies on a continuous elastic foundation, whose reaction at every point is proportional to the deflection (Winkler foundation). The equilibrium equation of the FG beams with piezoelectric layers located on a continuous elastic foundation subjected to axial compressive load is obtained from equilibrium equations by the addition of \( \eta w \) for the foundation reaction as

\[
(P - b A) \frac{d^2 w}{dx^2} + b A \left( \frac{d\phi}{dx} \right) + \eta w = 0, \quad A(\phi + \frac{dw}{dx}) + 2\alpha_1 V_T + 2D \left( \frac{d^2 \phi}{dx^2} \right) = 0
\]  

where \( \eta \) is the foundation coefficient.

**Stability Analysis:**

The boundary conditions for the pin-ended Timoshenko column are given by:

\[
w = \frac{d^2 w}{dx^2} = \frac{d\phi}{dx}, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L
\]

Substituting Eq. (10) into (9) and by equating power-law index to zero and neglecting the piezoelectric effect and foundation coefficient, the critical Engesser-Timoshenko buckling load of a homogeneous beam will be derived, that is:

\[
p_{cr} = \left( \frac{\pi}{L} \right)^2 \frac{b h Q_0}{12} \left( 1 + \frac{L}{\pi} \right)^2 \frac{12 Q_{11}}{bh^2 Q_{11}}
\]

The above equation has been reported by Wang and Reddy (Wang C.M., J.N. Reddy, 2000).

**Numerical Results:**

The mechanical buckling behaviors of simply supported functionally graded Engesser-Timoshenko beams with piezoelectric actuators located on a continuous elastic foundation are studied in this paper. The material properties of the beam are listed in Table 1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Piezoelectric layer</th>
<th>FGM layer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stainless steel</td>
<td>Nickel</td>
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<tr>
<td>Young's modulus</td>
<td>63</td>
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<tr>
<td></td>
<td>223.95</td>
<td></td>
</tr>
<tr>
<td>Poisson's ratio</td>
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<td>0.3</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Length</td>
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<td>0.3</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Thickness</td>
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<td>0.01</td>
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<tr>
<td></td>
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</tr>
<tr>
<td>Density</td>
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<td></td>
<td>9900</td>
<td></td>
</tr>
<tr>
<td>Piezo constant</td>
<td>17.6</td>
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</tbody>
</table>

The Poisson’s ratio is chosen to be 0.3 for both materials. The critical buckling loads for Bernoulli-Euler beam (BEB) and Engesser-Timoshenko beam (ETB) evaluated considering of \( h_0/h = 0.1 \), \( b/l = 1 \), \( L=1 \), \( V=10V \) and several values of dimensionless geometrical parameter \( h/L \) are shown in Fig. 2. It is seen that the critical buckling loads for Engesser-Timoshenko beam are generally lower than corresponding values of Bernoulli-Euler (Karami, 2008) beam. Fig. 3 demonstrates the buckling loads for functionally graded Engesser-Timoshenko beam. It is seen that the critical buckling loads for Engesser-Timoshenko beam increased with an increase of the ratio \( h/L \) and decreased with an increase of power-law index of constituent volume fraction.
Fig. 2: Critical Buckling Load of FG Smart Beam Versus $h/L$.

Fig. 3: Critical Buckling Load of FG Smart Beam Versus $h/L$ for $V=10v$.

Conclusion:

The stability of a functionally graded Engesser-Timoshenko beam with piezoelectric actuators located on a continuous elastic foundation subjected to axial compressive loads is studied. It is conclude that:

1. The piezoelectric actuators induce tensile piezoelectric force produced by applying negative voltages that significantly affect the stability of the functionally graded Engesser-Timoshenko beam with piezoelectric actuators.

2. The critical buckling loads of FG Engesser-Timoshenko beam are generally lower than corresponding values for the homogeneous Engesser-Timoshenko beam.

3. The critical buckling loads of FG Engesser-Timoshenko beam under axial compressive load generally increases with the increase of relative thickness $h/L$.

4. The accuracy of Engesser-Timoshenko beam theory is more than Bernoulli-Euler beam theory.

REFERENCES

