ADM Solution of Flow Field and Convective Heat Transfer over a Parallel Flat Plate and Comparison with the Forth Order Runge–Kutta Method

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Abstract: In this paper, the case of forced convection over a horizontal flat plate is presented and the Adomian decomposition method (ADM) is employed to calculate an approximation to the solution of the system of nonlinear differential equations governing on the problem. It has been attempted to show the potentials and wide-range applications of the Adomian decomposition method in comparison with the previous ones in solving heat transfer problems. The results are compared by the numerical technique so-called Forth Order Runge –Kutta method and Homotopy perturbation method (HPM). A clear conclusion can be drawn from the numerical results that the ADM provides highly precise numerical solutions for nonlinear differential equations. The effect of Adomian polynomials terms is considered on accuracy of the results. Also isotherm contours have been obtained in various condition of Re and Pr. Boundary layer thickness increases with Pr but decreases with Re and X velocity layer thickness decreases with Re too.

Keywords: Adomian Decomposition Method, Convection heat transfer, Flow field, Numerical Solution

INTRODUCTION

Most scientific problems such as heat transfer problems are inherently of nonlinearity. There are few phenomena in different fields of science occurring linearly. Therefore, these nonlinear equations should be solved using other methods. Some of them are solved using numerical techniques (Hasanpour et al. 2010) and some of them are solved using the analytical method of such as perturbation technique, HPM, HAM and etc. (Ganji et al. 2008, Ganji et al. 2009, Ganji et al. 2010, Ganji et al. 2011). The other well-known method is Adomian's decomposition method (ADM) (Adomian 1994, Adomian 1998, Ganji et al. 2011). The ADM was used to solve a wide range of physical problems. This method provides a direct scheme for solving linear and nonlinear deterministic and stochastic equations without the need for linearization and yields convergent series solutions rapidly. An advantage of this method is that, it can provide analytical or an approximated solution to a rather wide class of nonlinear (and stochastic) equations without linearization, perturbation, closure approximation, or discretization methods. Unlike the common methods are only applicable to systems with weak nonlinearity and small perturbation and may change the physics of the problem due to simplification. ADM gives the approximated solution of the problem without any simplification. Thus, its results are more pragmatic. During recent years, several researchers have tried to modify the ADM. Wazwaz (2001) developed a fast and accurate algorithm for solution of sixth-order boundary value problems. Jafari and Daftardar-Gejji (2006) modified ADM to solve a system of nonlinear equations. They obtained a series solution with faster accelerated convergence than the series obtained by the standard ADM. Luo (2005) proposed an efficient modification to ADM, namely two-step Adomian Decomposition Method (TSADM) that facilitated the calculations. Luo et al. (2006) revised ADM for cases involving inhomogeneous boundary conditions, using a suitable transformation. Zhang (2005) presented a modified ADM to solve a class of nonlinear singular boundary value problems, which start as nonlinear normal model equations in nonlinear conservative vibratory systems. Also several researchers have used the ADM to solve a wide range of physical problems in various engineering fields such as fluid flow and heat and mass transfer (Ganji et al. 2007). In this paper, the case of forced convection over a horizontal flat plate is presented and the Adomian decomposition method (ADM) is employed to calculate an approximation to the solution of the system of nonlinear differential equations governing on the problem. The goal of this study can be achieved by implementing the ADM to determine distribution of velocity and temperature boundary layer. Limit literature works on contours of temperature and velocity in analytical solutions. In this article we investigated these contours for showing results and effect of dimensionless number (Re and Pr) on problem clearly.
MATERIALS AND METHODS

**Adomian Decomposition Method (ADM):**

Consider equation \( F(u) = g(t) \), where \( F \) represents a general nonlinear ordinary or partial differential operator including both linear and nonlinear terms. The linear terms are decomposed into \( L+R \), where \( L \) is easily invertible (usually the highest order derivative) and \( R \) is the remains of the linear operator. Thus, the equation can be written as:

\[
Lu + Nu + Ru = g
\]

Where, \( Nu \) indicates the nonlinear terms. By solving this equation for \( Lu \), since \( L \) is invertible, we can write:

\[
L^{-1}Lu = L^{-1}g - L^{-1}Nu - L^{-1}Ru
\]

If \( L \) is a second-order operator, \( L^{-1} \) is a twofold indefinite integral, by solving Eq. (2) for \( u \), we get:

\[
u = A + Bt + L^{-1}g - L^{-1}Nu - L^{-1}Ru
\]

Where \( A \) and \( B \) are constants of integration and can be found from the boundary or initial conditions. ADM assumes the solution \( u \) can be expanded into infinite series as:

\[
u = \sum_{n=0}^{\infty} u_n
\]

(4)

Also, the nonlinear term \( Nu \) will be written as:

\[
Nu = \sum_{n=0}^{\infty} A_n
\]

(5)

Where \( A_n \) are the special Adomian polynomials. By substituting Eqs. (4) and (5) in Eq. (3), the solution can be written as:

\[
\sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n
\]

(6)

Where \( u_0 \) is identified as: \( A + Bt + L^{-1}g \) [3].

In Eq. (6), the Adomian polynomials can be generated by several means. Here we used the following recursive formulation:

\[
A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \right]_{\lambda=0}, n = 0,1,2,3,...
\]

(7)

Since the method doesn’t resort to linearization or assumption of weak nonlinearity, the solution is generated in the form of general solution and it is more realistic compared to the method of simplifying the physical problems.

**Basic Equations:**

Boundary layer flow over a horizontal flat plate is governed by the continuity and the Navier–Stokes equations. Under the boundary layer assumptions and a constant property assumption, the continuity and Navier–Stokes equations and energy transport equation become:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

(8)

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} + \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_w)
\]

(9)
From Eqs. (9) and (10), the solutions of the energy and momentum equations are coupled. However, the buoyancy force may be neglected if there is a pressure gradient at right angles to the gravitational force. Thus, in the case of the forced convection over a horizontal flat plate, the solution to the momentum equation is decoupled from the energy solution. The following dimensionless variables are introduced in the transformation:

$$\eta = \frac{y}{Re_x^{0.5}}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}$$

Where $\theta$ is non-dimensional form of the temperature and the Reynolds number is defined as:

$$Re_x = \frac{u_x x}{V}$$

By using Eqs. (8)-(13), the governing equations can be reduced to two equations where $F$ is a function of the similarity variable ($\eta$):

$$F'' + \frac{1}{2} FF = 0$$

$$\theta' + \frac{1}{2} \text{Pr}(F \theta') = 0$$

The reference velocity is the free stream velocity of forced convection. The boundary conditions for are obtained from the similarity variables. For the forced convection case (1995):

$$F(0) = 0, F'(0) = 0, F'(\infty) = 1$$

$$\theta(0) = 1, \theta(\infty) = 0$$

If ADM applies on the Eqs. (12)-(16) in operator form as:

$$LF = \frac{1}{2} FF' = 0$$

$$L\theta = \frac{1}{2} \text{Pr}(F \theta') = 0$$

Where the differential operator $L$ is given by $L = \frac{d^3}{d\eta^3}$ and have been assumed the inverse of the operator $L$ exists and it can be integrated from 0 to $\eta$, i.e. $L^{-1} = \int_0^\eta \int_0^\eta (\cdot) d\eta d\eta d\eta$.

If $L^{-1}$ Operates on Eqs. (18) and (19), yields $L^{-1} LF = L^{-1}(\frac{1}{2} FF')$ and $L^{-1} L\theta = \frac{1}{2} \text{Pr}(F \theta')$. Then, we have:

$$F(\eta) = F(0) + \eta \ F'(0) + \frac{\eta^2}{2} \ F''(0) + L^{-1}(\frac{1}{2} FF')$$

$$\theta(\eta) = \theta(0) + \eta \ \theta'(0) + \frac{\eta^2}{2} \ \theta''(0) + L^{-1}(\frac{1}{2} \text{Pr}(F \theta'))$$
From the boundary conditions (Eqs. (16) and (17)) and taking \( F'(0) = \alpha \) and \( \Theta'(0) = \beta \), ADM solution can be obtained by:

\[
F(\eta) = \alpha \frac{\eta^2}{2} + L^{-1}\left(-\frac{1}{2}FF'\right) \tag{22}
\]

\[
\Theta(\eta) = 1 + \beta \eta + L^{-1}\left(-\frac{1}{2}\text{Pr}(F'\theta')\right) \tag{23}
\]

ADM is introduced in the following expression:

\[
F(\eta) = \sum_{n=0}^{\infty} F_n(\eta) \tag{24}
\]

\[
\Theta(\eta) = \sum_{n=0}^{\infty} \Theta_n(\eta) \tag{25}
\]

The ADM is defined as the nonlinear function \( G(F(\eta)) \) by an infinite series of polynomials:

\[
G(F(\eta)) = \sum_{n=0}^{\infty} A_n \tag{26}
\]

\[
\Theta(\Theta(\eta)) = \sum_{n=0}^{\infty} B_n \tag{27}
\]

Adomian polynomials \( A_n \) represent the nonlinear term \( G(F(\eta)) \) and can be calculated from Eq. (7). By substituting Eqs. (24) - (27) into Eqs.(22) and (23) yields:

\[
\sum_{n=0}^{\infty} F_n(\eta) = \alpha \frac{\eta^2}{2} + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{28}
\]

\[
\sum_{n=0}^{\infty} \Theta_n(\eta) = \beta \eta + L^{-1}\left(\sum_{n=0}^{\infty} B_n\right) \tag{29}
\]

To determine the components of \( A_n \) and \( F_n(\eta) \), \( F_0(\eta) \) was defined from the boundary condition of Eqs.(16) and (17) at \( \eta = 0 \):

\[
F_0(\eta) = \alpha \frac{\eta^2}{2} \tag{30}
\]

\[
\Theta_0(\eta) = 1 + \beta \eta \tag{31}
\]

For determination of the other components of \( F(\eta) \) and \( \Theta(\eta) \), we have:

\[
F_{n+1}(\eta) = L^{-1}(A_n) \quad n = 0,1,2,\ldots \tag{32}
\]

\[
\Theta_{n+1}(\eta) = L^{-1}(B_n) \quad n = 0,1,2,\ldots \tag{33}
\]

By using Eq (7), we obtain the following terms of Adomian polynomials \( A_n \):
and this calculation can be continued until \( A_n \).

And from Eq(28), for determining of \( F_n(\eta) \), we have:

\[
A_0 = -\frac{1}{2} F_0 F_0' \\
A_1 = \frac{1}{2} (F_0 F_0' + F_0 F_1') \\
A_2 = -\frac{1}{2} (F_0 F_0' + F_1 F_1' + F_0 F_2') \\
A_3 = -\frac{1}{2} (F_0 F_0' + F_1 F_1' + F_2 F_2' + F_0 F_3') \\
A_4 = -\frac{1}{2} (F_4 F_0' + F_1 F_1' + F_2 F_2' + F_3 F_3' + F_0 F_4')
\]

(34)

We use:

\[
F = \sum_{n=0}^{\infty} F_n = F_0 + F_1 + F_2 + F_3 + F_4 + ...
\]

(36)

And similar to above:

\[
B_0 = -\frac{Pr}{2} \theta_0 (0.1803 \eta^2 - 0.0005 \eta^5 + 0.0000032 \eta^8 - 1.9847 \times 10^{-8} \eta^{11} + 1.2187 \times 10^{-10} \eta^{14} + 7.368 \times 10^{-13} \eta^{17})
\]

\[
B_1 = -\frac{Pr}{2} \theta_1 (0.18028 \eta^2 - 0.000542 \eta^5 + 0.000032 \eta^8 - 1.984 \times 10^{-8} \eta^{11} + 1.22 \times 10^{-10} \eta^{14} + 7.369 \times 10^{-13} \eta^{17})
\]

\[
B_2 = -\frac{Pr}{2} \theta_2 (0.18026 \eta^2 - 0.000542 \eta^5 + 0.000032 \eta^8 - 1.984 \times 10^{-8} \eta^{11} + 1.22 \times 10^{-10} \eta^{14} + 7.369 \times 10^{-13} \eta^{17})
\]

(37)

And from Eqs. (29), for determining of \( \theta_n(\eta) \), we have:

\[
\theta_0 = 1 + \beta \eta \\
\theta_1 = \beta \Pr (1.0773 \times 10^{-5} \eta^2 - 2.539 \times 10^{-13} \eta^{14} + 6.362 \times 10^{-13} \eta^{14} - 1.776 \times 10^{-10} \eta^{14}) + 6.448 \times 10^{-8} \eta^{7} - 0.00751 \ln \eta^4)
\]

\[
\theta_2 = \beta \Pr (1.0773 \times 10^{-5} \eta^2 - 2.539 \times 10^{-13} \eta^{14} + 6.362 \times 10^{-13} \eta^{14} - 1.776 \times 10^{-10} \eta^{14}) + 6.448 \times 10^{-8} \eta^{7} - 0.00751 \ln \eta^4)
\]

(38)

We use:

\[
\theta = \sum_{n=0}^{\infty} \theta_n = \theta_0 + \theta_1 + \theta_2 + ... \theta_n
\]

(39)
Clarification of Numerical Method:

The type of current problem is Boundary Value Problem and Maple software like appropriate software is used as a numerical solver. In this study the midpoint method has been used. There are two major considerations when choosing a method for a problem. The trapezoid method is generally efficient for typical problems but the midpoint method is so capable of handling harmless end-point singularities that the trapezoid method cannot. The midpoint method, also known as the Runge-Kutta method, improves the Euler method by adding a midpoint in the step which increases the accuracy.

RESULTS AND DISCUSSIONS

This study considers the steady state boundary layer flow. ADM have been used for calculation the momentum and energy equations of the forced convection over a horizontal flat plate. The series for \( F \), \( F' \) (the velocity profile) and temperature field \( \theta \), evaluated and are presented in Figs. (1) – (12). Figures (1) - (3) show the effect of number of series’ term in correctness of ADM solution, For \( F \) and \( F' \) (velocity field). It is obvious from this figure that when the terms of series is expand, the convergence for velocity distribution is better. In Fig. (2) The variations of the Prandtl number (Pr) on the temperature (\( \theta \)) are shown.

Fig. 1: Comparison \( F (\eta) \) and \( F'(\eta) \) between 3 terms and 5 terms in ADM series, Pr=1

Fig. 2: Comparison of temperature between various Pr number in temperature distribution

It can be seen that the temperature (\( \theta \)) decrease with Pr. Because velocity profile is independent of Pr number, therefore the same curve is not plotted for \( F' \). In order to verify the accuracy of the current study illustrates the comparison between ADM method and homotopy perturbation method (2007) and numerical solution are shown in Table (1). The results prove that the answers for velocity and temperature distributions have a good agreement with another method and particularly with numerical solution.

Also the results for analytical solutions drown in 2-D graph for comparing effect of Re and Pr number. Effect of Re number on isotherm contour are shown in Figs. (3 and 4).
Table 1: The results of ADM, HPM [24] and Numerical methods for $F(\eta)$, $F'(\eta)$ and $\theta(\eta)$

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<th>HPM</th>
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Fig. 3: Contour isotherms for Re/\(x=0.1, \text{Pr}=1\)

Boundary layer thickness decreases with Re. Boundary layer thickness increases with Pr, in smaller isotherm line ($\theta$) this difference have been bigger than larger isotherm line. These results are shown in Fig. 4.

Fig. 4: Effect of Pr on Contour isotherms for Re/\(x=0.1, \text{Pr}=0.72\) (dashed lines), 1.5 (solid lines)

Conclusions:

The main aim of this paper is the Application of Adomian Decomposition Method to boundary layer flow and convection heat transfer over a horizontal flat plate. Graphical results are presented to examine the effects of the Prandtl number (\(Pr\)), the extension of polynomial series of Adomian Decomposition method on the \(F, F'(\eta)\) (velocity profile) and temperature profiles (\(\theta\)). We obtained isotherm contours in various condition of Re and Pr and are shown Boundary layer thickness increases with Pr but decreases with Re. according to the analytical solution X velocity layer thickness decreases with Re.

REFERENCES
