Getting Smaller Dixon Dialytic Matrix Through Polynomials System Manipulating

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Abstract: Resultant is the result of eliminating the variables from a system of polynomials. Compared with other matrix based methods such as Sylvester, Macaulay and Sturmfels et al, Dixon formulation is one of the most efficient tools to compute resultants. Previously it was thought that the Dixon resultant formulation defines a completely different class of resultant formulations. Recently a number of mixed constructions have been proposed where some parts of the matrix are of Sylvester type and some parts are of the Dixon type. Dixon Dialytic matrix is one such formulation. These constructions produce smaller matrices at the expense of more complicated matrix entries. Dixon Dialytic method, similar to other resultant methods, are subjected to producing of extraneous factors along with the exact resultant, however; recognition of these factors takes a long process and sometimes the presence of them can make computing the projection operator, impractical. It is proposed a new efficient method for reducing or eliminating the number of extraneous factors from the Dixon Dialytic resultant construction. In the Dixon Dialytic formulation, there is a parameter which is used to further reduce the number of the extraneous factors. Experimental results suggest success of the new algorithm. The scope of this research falls under computational algebra, algebraic geometry and elimination theory.

Key words: Elimination Theory, Resultant, Dixon Construction, Extraneous Factors.

INTRODUCTION

Resultant theory has a long mathematical story, starting with the resolution of linear polynomials system. The first explicit construction so called Resultant of two univariate polynomials, proposed by L. Euler et al., (1965) and E. B’ezout (1779) was followed by the well known Dialytic method of Sylvester (1853). The generalizations of this method to multivariate polynomials appeared after 1860 at which point intensive studies were initiated by Salmon (1924), Macaulay (1902), Dixon (1908) and Van Der Waerden (1950) in this area. But after this point it started to be forgotten, and until 1990’s, it did not receive much attention by mathematicians. It could be due to the fact that the structure of their methods required computations which were based on long computing processes. Therefore the application of the Resultant theory has not been realized until recently.

Resultant, which is also called Eliminant in the classical literature (G. Salmon, 1902) is the result of variable eliminating from given polynomials system. In general, from \( d + 1 \) polynomials, \( d \) variables can be eliminated to obtain a single polynomial in the remaining variables. This allows establishing direct dependencies among the remaining variables. All multivariate Resultant methods, except in some special cases, compute a nontrivial multiple of the Resultant. In other words, they compute the Resultant possibly multiplied with some extraneous factors. These extraneous factors are undesirable and they create problems in certain applications. There is a need to develop or construct techniques that can eliminate, or at least reduce the number of extraneous factors from all the multivariate Resultant methods.

There are two main matrix based methods to solve a polynomials system which are the Sylvester and B’ezout-Cayley method. Dixon Dialytic, is B’ezout-Cayley based which can be considered as a dense and small matrix producer (T. Saxena, 1997), (E. Bezout, 1779). Dixon Dialytic method likewise other resultant methods are subjected to extraneous factors. They also present such projection operators that include resultants and some surplus troublemaker’s factors.

Dixon Dialytic Resultant:

Based on the Dixon resultant formulation discussed in (S. Karami et al., 2010) this construction can be viewed as a generalization of the Sylvester construction for the univariate case to the multivariate case.

Let \( g \in \mathbb{C}[x,c] \) be an arbitrary polynomial. Given a polynomial system \( \mathcal{F} = \{f_0, f_1, \ldots, f_d\} \subseteq \mathbb{C}[x,c] \), let the Dixon polynomial:

\[
\theta_i(g) = \theta(f_0, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_d)
\]
Then $\theta_i(g)$ is the Dixon polynomial of polynomial system $\mathcal{F}$ where the $i^{th}$ polynomial is replaced by $g$. Easily it is shown in (A. Chtcherba and D. Kapur, 2002):

$$g \theta(f_0, ..., f_d) = \sum_{i=0}^{d} f_i \theta_i(g)$$

That $g$ is an arbitrary polynomial. The above identity was already used in (A. Chtcherba and D. Kapur, 2000) as well (J.P. Cardinal and B. Mourrain, 1996) to show that the Dixon polynomial is in the ideal of the original polynomial system when $g = 1$.

If we let $g$, the support of $\theta$, then we can write:

$$g \theta(f_0, f_1, ..., f_d) = \sum_{a \in g} g_a x^a \theta(f_0, f_1, ..., f_d)$$

Since our goal is to create smaller Dixon matrices, we can work on minimizing the $\theta_i(f_0, ..., f_{i-1}, g_a x^a, f_{i+1}, ..., f_d)$s, then it is not useful to consider parameter $g$ as a general polynomial when single monomial is suffice.

In the bilinear form:

$$\theta_i(g) = X_i^T \Omega_i(g) X_i$$

Where $\Omega_i(g)$ is the Dixon matrix of $\{f_0, ..., f_{i-1}, g, f_{i+1}, ..., f_d\}$. By expressing $\theta_i(g)$ in term of $\Omega_i(g)$, we have:

$$\theta_i(g) f_i = \left( X_i^T \Omega_i(g) X_i \right) f_i = \left( X_i^T \Omega_i(g) \right) (X_i f_i)$$

Thus, we can construct a Dialytic matrix by using monomial multipliers $X_i$ for $f_i$.

In Sylvester construction we have some form like:

$$\mathcal{F}' = \begin{pmatrix} X_{a_0 f_0} \\ X_{1 f_1} \\ \vdots \\ X_{d f_d} \end{pmatrix}$$

If we let $\mathcal{M}_{\mathcal{F}'}$, the Sylvester matrix and $Y = \bigcup_{i=0}^{d} X_i$, we can have $\mathcal{F}' = \mathcal{M}_{\mathcal{F}'} \times Y$.

Now we can rewrite the formula for the Dixon Dialytic polynomial as:

$$g \theta(f_0, ..., f_d) = X \Theta g X$$

$$= \sum_{i=0}^{d} \theta_i(g) f_i$$

$$= \sum_{i=0}^{d} \left( X_i^T \Omega_i(g) \right) (X_i f_i)$$

$$= \tilde{Y} \left( \Theta_0(g) : \Theta_1(g) : ... : \Theta_d(g) \right) \begin{pmatrix} X_{a_0 f_0} \\ X_{1 f_1} \\ \vdots \\ X_{d f_d} \end{pmatrix}$$

$$= \tilde{Y} \left( T \times M_{\mathcal{F}'} \right) Y = \tilde{Y} \Theta Y$$

Where $\tilde{Y} = \bigcup_{i=0}^{d} X_i, T = [\Theta_0(g) : \Theta_1(g) : ... : \Theta_d(g)]$ And $\Theta = T \times M_{\mathcal{F}'}$. Therefore:

$$X \Theta g X = \tilde{Y} \Theta Y$$
Note that the monomials of \( gX \) are a subset of \( Y \), and \( \overline{X} \subseteq \overline{Y} \); therefore, \( \Theta \) and \( \Theta' \) are the same matrices except for \( \Theta \) having some extra zero in rows and columns. The Cauchy-Binet formula can be useful tool to find the determinant of \( \Theta' \) to find the projection operator.

The sub Dixon Dialytic matrices, used in the construction of a Dixon Dialytic matrix, are related to the monomials of the Dixon polynomial, which also label the columns of the Dixon matrix. For a given polynomial \( f_i \), its sub Dixon Dialytic matrix which is generated by using a polynomial parameter \( g \) with the support \( G \) is obtained from \( \theta_i(f_0, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_d) \). The support of the Dixon polynomial for such polynomial system is \( \langle \mathcal{A}_0, \ldots, \mathcal{A}_{i-1}, G, \mathcal{A}_{i+1}, \ldots, \mathcal{A}_d \rangle \) which is denoted as \( \mathcal{A}(i, G) \).

**Theorem 2.1:**

For given \( \mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_d \rangle \) as defined above and a support \( G \subseteq \bigcap_{i=0}^{d} \mathcal{A}_i \):

\[
\Delta_\mathcal{A} = \bigcup_{i=0}^{d} \Delta_{\mathcal{A}(i, G)}
\]

**Proof:**

A general proof of this theorem is presented in (A. Chtcherba and D. Kapur, 2002).

By using above theorem we can prove that the size of a Dixon matrix is at least as big as the size of the largest sub Dixon Dialytic matrix of the corresponding Dixon Dialytic matrix.

Let:

\[
\Phi_i(G) = \left| \Delta_{\mathcal{A}_0, \ldots, \mathcal{A}_{i-1}, G, \mathcal{A}_{i+1}, \ldots, \mathcal{A}_d} \right|, \quad \Phi(G) = \sum_{i=0}^{d} \Phi_i(G)
\]

i.e., \( \Phi_i(G) \) is the number of columns of Dixon Dialytic matrix form for each polynomial \( f_i \), and \( \Phi(G) \) is the number of columns of the Dixon Dialytic matrix constructed using \( g \) with support \( G \) as the parameter.

**Theorem 2.1:**

For a generic polynomial system \( F \), the size of the Dixon matrix is at least as big as the size of the largest Dixon Dialytic sub matrix for the Dixon Dialytic matrix, i.e.:

\[
\text{Size}(\Theta) = |\Delta_\mathcal{A}| \geq \max_{i=0}^{d} \Phi_i(G) \text{ whereas } G \subseteq \bigcap_{i=0}^{d} \mathcal{A}_i
\]

**Proof:**

A good extended proof can be find in (A. Chtcherba and D. Kapur, 2002).

Then in mixed generic cases, the Dixon Dialytic matrix doesn’t have bigger extraneous factor’s degree. The above relation becomes equality in the unmixed case, when \( \mathcal{A}_i = \mathcal{A}_j \) for all \( 1 \leq i \neq j \leq d \).

**Minimizing of Dixon Dialytic Matrix:**

Minimizing the size of the Dixon matrix was the goal of (S. Karimi et al., 2010). A slightly modified method is needed for the Dixon Dialytic matrices to minimize the sizes of each Dixon polynomials \( \theta_i(x^\alpha) \), i.e., \( \Phi_i(\mathcal{A}_i, t) \) for each \( i \) which also involves choosing the parameter \( \alpha \), where \( \mathcal{A}_i \) is support of the parameter and \( t \) is the shifting support.

Since \( \Phi_i(\mathcal{A}_i, t) \) represents the number of columns corresponding to the polynomials \( x^t f_i \) in the Dixon Dialytic matrix, the goal is to find \( t \) and \( t = (t_0, t_1, \ldots, t_d) \) such that \( \Phi_i(\mathcal{A}_i, t) \) be minimized, i.e., the size of the entire Dixon Dialytic matrix will be minimized to decrease the degree of the extraneous factors part of projection operator.

Since terms \( \Phi_i(\mathcal{A}_i, t) \) are interdependent, an optimal \( t \) and \( \alpha \) for \( \Phi_i(\mathcal{A}_i, t) \) might be quite bad for \( \Phi_j(\mathcal{A}_j, t) \), when \( i \neq j \), thus it is extra problem to minimize the sum for \( \Phi_i(\mathcal{A}_i, t) \). Following observations can help to choose \( t \) and \( \alpha \) to minimizing the \( \Phi(\mathcal{A}_i, t) \).

Let \( \Delta_{\mathcal{A}+t}(i, \mathcal{A}) \) is the support of Dixon Dialytic polynomial with parameter support \( \{\alpha\} \) then, by Theorem 2.1.1:

\[
\bigcup_{i=0}^{d} \Delta_{\mathcal{A}+t}(i, \mathcal{A}) \subseteq \Delta_{\mathcal{A}+t} \text{ and therfor for all } i, \quad \Phi_i(\mathcal{A}_i, t) \leq |\Delta_{\mathcal{A}+t}|
\]
Since it is difficult to minimize the size of $\Delta_{A+h(t)}$, that is $\Phi_t(\{a\}, t)$, we will minimize the $|\Delta_{A+h+t}|$.

As soon as standing $t$ on fixing point by using the above procedure to minimize $|\Delta_{A+h+t}|$, search for a monomial with exponent $\alpha$ for constructing the Dixon dialytic matrix will be start. A direct proposition detected by considering the theorems 2.1.1 and 2.1.2 can be:

$$G \subseteq \bigcap_{i=0}^{d} A_i$$

Then the best guess for finding $\alpha$ that is the support of parameter $g$ can be:

$$\alpha \in \text{SupportHull} \left( \bigcap_{i=0}^{d} (t_i + A_i) \right)$$

**Example 3.1:**

Consider the following polynomial system:

$$F = \begin{cases}
  f_0 &= a_0 + a_1x^2 + a_2x^3y^6 + a_3x^7y^6 \\
  f_1 &= b_0x + b_1y^7 + b_2x^2y^9 + b_3x^3y^9 \\
  f_2 &= c_0 + c_1x^2y^5 + c_2x^3y^4
\end{cases}$$

$x, y$ are variables and $a_i, b_i, c_i$ are parameters. The supports are:

$A_0 = \{(0,0), (2,0), (3,6), (7,6)\}$, $A_1 = \{(1,0), (0,7), (2,9), (3,9)\}$

And; $A_2 = \{(0,0), (2,5), (8,4)\}$

With support hulls:

**Fig. 3.1:** Supports hulls for the polynomials in Ex 3.1.

At the end of optimizing process presented in (S. Karimi et al., 2010) we it is found a figure like below of the supports hulls. Then, according to above illustrations, $\alpha$ can be chosen from the set \{\{(10,12), (11,12), (11,13), (11,14)\}\}. Each point which earns the best Dixon matrix size can introduce the optimal $\alpha$ for the Dixon Dialytic construction.

**Fig. 3.2:** Supports hulls for the polynomials in Ex 3.1 after applying optimizing method and choosing the $\alpha$. 
The table below shows the size of Dixon Dialytic matrix and its sub matrices in any choice of $g$. Obviously Dixon Dialytic matrix gets its best size when $g = x^{10} y^{12}$.

| $g$      | $|\mathcal{P}|$ | $|\mathcal{G}|$ | $|\mathcal{A}|$ | $|\mathcal{A'}|$ |
|----------|----------------|----------------|---------------|----------------|
| $(10,12)$ | 75 x 77        | 55 x 53        | 65 x 65       | 75 x 195       |
| $(11,12)$ | 77 x 79        | 57 x 55        | 68 x 68       | 77 x 202       |
| $(11,13)$ | 75 x 77        | 56 x 54        | 67 x 66       | 75 x 197       |
| $(11,14)$ | 75 x 77        | 55 x 53        | 66 x 66       | 75 x 196       |

**Complexity Analysis:**

The complexity of minimizing direction, for the system of polynomials $\mathcal{F} = \{f_0, f_1, ..., f_d\}$, can be earned by recognition of research area along with the cost of receiving better Dixon matrix in each phase of illustrated reiterative method which presented in [9] as $O(kd^5n^4)$.

In Dixon Dialytic, we don’t have a big difficulty, because minimizing the Dixon matrix guaranties optimization of the Dixon Dialytic, as illustrated in section 2. In the worse case, we have $n$ choices for $\alpha$, where $n = |\bigcup_{i=1}^{d} \mathcal{A}_i|$, and then the total complexity for finding best Dixon Dialytic matrix can be:

$O(kd^5n^{d+1})$

**Conclusion:**

In this paper, a method for reducing the number of extraneous factors in the projection operator as a result of deriving the Dixon Dialytic method on multivariate polynomials systems is presented. As a matter of fact the relation between the process of optimizing of the Dixon matrix and optimizing of the Dixon Dialytic matrix is illustrated in details.

In the Dixon Dialytic method, which is a kind of Sylvester type construction of extended Dixon formulation, there is a parameter which is specified to attain a smaller resultant matrix. At the end, the complexity analysis for the method is presented.

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