He’s Homotopy Method for Flow of a Conducting Fluid through a Porous Medium

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Abstract: In the present work an approximate solution is presented for Couette flow through a porous saturated parallel plates channel subjected to the Lorentz force for an electrically conducting fluid. Applying Homotopy Perturbation Method (HPM), dimensionless velocity profile has been obtained for a range of the key parameters considered in this study. Results are then compared with an independent numerical solver that uses fourth order Runge–Kutta scheme to observe an excellent agreement. The merit of solution is presenting result with good accuracy just by two approximations due to suitable choice of linear part.

Key words: Homotopy Perturbation Method, Darcy-Brinkman porous media, Runge-Kutta Scheme, Collocation Method.

INTRODUCTION


Finding an exact analytical solution for flow of conducting fluid through a Darcy-Brinkman porous media is easy in the light of (Kaviany, M., 1985). But when it comes to consider effect of non-linear drag term, the problem becomes more complex and finding an analytical exact solution is not an easy task. This paper aims at applying Homotopy Perturbation Method (HPM), a powerful method of solving both linear and non-linear differential equations, as introduced by (He, J.H., 1999; He, J.H., 2003) and further applied by (Ganji, D.D. and Sadighi, 2008; Biazar, J., et al., 2009; Seyf, H. and Layeghi, M., 2010) to a number of problems, to present an analytical solution for dimensionless velocity distribution for Couette flow of an electrically conducting fluid through a porous-saturated PPC affected by Lorentz force. This work offers an inherent advantage over HAM-based results as the leading two terms in HPM are accurate enough to give a very appropriate approximation of the exact solution. Thus, the CPU usage for HPMs is much lower than HAMs.

Governing Equations:

The momentum balance for flow of a viscous and incompressible fluid through a porous PPC can be written as:
\[ \frac{dp}{dx} + \mu_{\text{eff}} \frac{d^2 u}{dy^2} - \frac{\mu u}{K} \frac{c_f \rho}{K^{1/2}} u^2 = 0 \]  

By adding the effect of Lorentz force to our momentum equation, similar to (Zhao, B.Q., et al., 2010), it takes the following form:

\[ \frac{dp}{dx} + \mu_{\text{eff}} \frac{d^2 u}{dy^2} - \frac{\mu u}{K} \frac{c_f \rho}{K^{1/2}} u^2 + \frac{\pi j_0 M_0}{8} \exp(-\frac{\pi}{h} y) = 0 \]  

with boundary conditions \( u(0)=0 \), \( u(h)=u_w \) for unidirectional Couette flow.

The dimensionless form of the momentum equation, following the use of the channel opening and \( u_w \) as the length and velocity scale, takes the following form by elucidating \( U=u/u_r \) and \( Y=y/h \)

\[ A + \frac{d^2 U}{dY^2} - s^2 U - FU^2 + Q \exp(-BY) = 0, \]  

\[ A = -\frac{h^2}{\mu_{\text{eff}} u_r} \frac{dp}{dx}, Q = \frac{\pi j_0 M_0 h^2}{8u_r \mu_{\text{eff}}}, B = \frac{\pi h}{a}, s^2 = \frac{\mu h^2}{\mu_{\text{eff}} K^{1/2}}, F = \frac{c_f \rho u_r h^2}{\sqrt{K} \mu_{\text{eff}}} \]

For the Couette flow we have \( u_r = u_w \), then Eq.3 should be solved subject to the following boundary conditions:

\[ \begin{cases} U(0) = 0 \\ U(1) = 1 \end{cases} \]  

Homotopy Perturbation Method:

Let us consider following differential equations:

\[ A(r) = f(r) \quad r \in \Omega \]  

With the following boundary conditions:

\[ B(u, \frac{\partial u}{\partial n}) = 0 \quad r \in \Gamma \]  

where \( f(r) \) is a known analytical function, \( A \) is a general differential operator, \( B \) is a boundary operator, \( \frac{\partial}{\partial n} \) is differentiation along the normal vector drawn outward from \( \Omega \), and \( \Gamma \) is boundary of the domain \( \Omega \).

The operator \( A \) can be divided in two linear (L) and nonlinear (N) parts, So, Eq. (1) can be rewritten as follows:

\[ L(u) + N(u) - f(r) = 0 \]  

The Homotopy perturbation structure is:

\[ D(t,q) = (1-q) \left[ \frac{L(t)}{L(u_0)} \right] + q \left[ A(t) - f(r) \right] = 0, r \in \Omega, q \in [0,1] \]  

Where

\[ \frac{dp}{dx} + \mu_{\text{eff}} \frac{d^2 u}{dy^2} - \frac{\mu u}{K} \frac{c_f \rho}{K^{1/2}} u^2 = 0 \]  

\[ \frac{dp}{dx} + \mu_{\text{eff}} \frac{d^2 u}{dy^2} - \frac{\mu u}{K} \frac{c_f \rho}{K^{1/2}} u^2 + \frac{\pi j_0 M_0}{8} \exp(-\frac{\pi}{h} y) = 0 \]  

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Where
\( t(r,q) = \Omega \times [0,1] \rightarrow R \) \hspace{1cm} (10)

In Eq. (4), \( q \in [0,1] \) is an embedding parameter, which increases from 0 to 1 and \( u_0 \) is the initial approximation of Eq. (1) that satisfies boundary conditions. By Eq. (9), it easily follows that:

\[
D(t,0) = L(t) - L(u_0) = 0 \quad (11)
\]

\[
D(t,1) = A(t) - f(r) = 0 \quad (12)
\]

Changing values of \( q \) form zero to unity lead to changing \( t(r,q) \) from \( u_0(t) \) to \( u(r) \). In topology, this is called deformation and \( L(t) - L(u_0) \) and \( A(t) - f(r) \) are called homotopic. According to HPM, we can first use the embedding parameter \( s \) as a “small parameter”, so by applying the perturbation technique, we assume that the solution of Eq. (9) can be written as a power series in \( s \), as following:

\[
t = t_0 + t_1 q + t_2 q^2 + t_3 q^3 + \ldots
\] \hspace{1cm} (13)

The best approximation for Eq. (1) can be readily obtained by setting \( q = 1 \) as follows:

\[
u = \lim_{s \to 1} t = t_0 + t_1 + t_2 + t_3 + \ldots \] \hspace{1cm} (14)

The series (14) is convergent for most cases. However, the convergent rate depends on the nonlinear operator \( A(t) \) and suitable choice of linear term.

**Method of Solution:**

To solve Eq. (3) by HPM, the linear operator which is the linear part of the Eq. (3) is determined as

\[
L(g) = \frac{d^2 U}{dY^2} - Us^2 \] \hspace{1cm} (15)

Where \( g \) is an auxiliary function.

The next step is to guess an arbitrary initial approximation that satisfies the boundary conditions as follows:

\[
U_{ini} (Y) = 2Y^2 - Y \] \hspace{1cm} (16)

Where the subscript “ini” refers to an initial approximation for the solution of Eq. (3).

According to Eq. (4) and HPM, the following homotopy equation would be constructed

\[
D(g,q) = \left(1 - q \right) \left[ \frac{d^2 U}{dY^2} - s^2 U - \frac{d^2 U_{ini}}{dY^2} - s^2 U_{ini} \right] + q \left[ A + \frac{d^2 U}{dY^2} - s^2 U - FU^2 + Q \exp(-BY) \right] = 0 \] \hspace{1cm} (17)

Implementing the perturbation technique, we assume that the solution to Eq. (17) can be written as a power series in \( p \), that is:

\[
g = g_0 + g_1 q + g_2 q^2 + g_3 q^3 + \ldots \] \hspace{1cm} (18)

Hence, substituting Eq. (18) into Eq. (17) and equating the coefficient with identical power of \( p \) we obtain a system of equations with \( n+1 \) differential equations to be solved simultaneously (when \( n \) is the order of \( p \), i.e.

**Zeroth-order:**

\[
\frac{d^2}{dY^2} g_0(Y) - s^2 g_0(Y) - 4 + s^2 (2Y^2 - Y) = 0 \] \hspace{1cm} (19a)
First order:

\[ -s^2 \alpha g_1(Y) + \frac{d^2}{dY^2} g_1(Y) + 4 - s^2 (2Y^2 - Y) + Q e^{-BY} - A - Fg_0(Y) = 0 \]  

(19b)

And the boundary conditions are: \( g_i(0) = 0, g_i(1) = 1 \) for \( i = 0 \) and \( g_i(0) = 0, g_i(1) = 0 \) for \( i > 0 \).

Solving Eqs. (19a-b) subject to corresponding boundary conditions, results in.

\[
g_a(Y) = 2Y^2 - Y
\]

\[
g_i(Y) = \frac{-(-2e^{-s^2-1})F s^4 - e's^6yB^2 + e^{-3}s^6A - 96e'^2 FB^2 + ...}{(-s^2 + B^2)(e'' - e')s^2}
\]

(20a-b)

Finally, by summing up the results, with \( q = 1 \), we write the velocity profile as:

\[
U(Y) = \sum_{i=0}^{1} g_i(Y)
\]

(21)

Equation (21) is the analytical solution to the momentum equation, Eq. (3).

Having solved for the velocity field, the temperature profile can now be obtained by integrating Eq. (10) and implementing the boundary conditions (11). At last, Eq. (12) can be solved to get the Nusselt number. A fourth order Runge–Kutta scheme solver has been applied to solve the momentum equation to check the accuracy of the solution.

RESULTS AND DISCUSSIONS

Both theoretical and numerical results are presented in this section. As a common trend in all figures the results from the two solvers are in good agreement and cover all range of \( s \) as the most influential parameters. The effects of different parameters, being \( A, B, Q, \) and \( s \), on fluid flow through the porous PPC are parametrically investigated.

Figure 3 demonstrates the effect of the porous medium shape parameter on the velocity profile for the case of \( A=0 \), i.e. negligible pressure gradient. As seen there is good agreement between FDM ans HPM solutions.

**Fig. 1:** Effect of porous media shape parameter on velocity profile when \( Q=10, B=\pi, A=0, F=1 \).

Figure 4 is presented to show the effect of \( Q \) on velocity distribution. As seen increasing \( Q \) the Lorentz force increases the fluid velocity in the core. As the fluid velocity at the two plates is fixed, this increase in velocity has to inflate the velocity profile far away from the walls and in the limit a \( S \) shape velocity profile can be observed as \( Q \) rises beyond 50.
Fig. 2: Effect of $Q$ on velocity profile when $x^2=30$, $B=\pi$, $A=0$, $F=1$.

Figure 5 illustrates the effects of $B$ on the velocity profiles. As expected, with a fixed $Q$ value, increasing $B$ levels down the velocity profiles.

Fig. 3: Effect of $B$ on velocity profile when $x^2=30$, $Q=10$, $A=0$, $F=1$.

Conclusion:
Lorentz force effects on fluid flow through a porous medium are examined theoretically based on HPM. A numerical simulation is also provided for comparison purpose. It was observed that a proper choice of the linear part in HOM solution leads to converging of the approximation with summing up two orders which leads to less CPU usage and time saving in computations.

Nomenclature:
- $a$: width of magnets and electrodes
- $A$: dimensionless applied pressure gradient, $-\frac{h^2}{\mu_{\text{eff}}} \frac{dp}{dx}$
- $B$: $\frac{\pi h}{a}$
- $c_F$: form drag coefficient
specific heat at constant pressure

dimensionless form drag coefficient

channel opening

applied current density

thermal conductivity

permeability

magnetization of the permanent magnet

Nusselt number

Chandrasekhar-Darcy number

Shape parameter

dimensionless velocity \( y \) \( (U=u/ur) \)

velocity of the upper plate

mean velocity

normalize velocity \( \overline{U} = \frac{U}{u} \)

filtration velocity

reference velocity

longitudinal coordinate

transverse coordinate

dimensionless transverse coordinate \( (Y=y/h) \)

fluid viscosity

effective viscosity

fluid density

REFERENCES


