The Spectrum of Classical Prime Subsemimodules

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Abstract: Let \( S \) be a commutative semiring and \( M \) an \( S \)-semimodule. A proper subsemimodule \( N \) of \( M \) is called a classical prime subsemimodule, if for any \( a,b \in S \) and \( m \in M \), \( abm \in N \) implies that \( am \in N \) or \( bm \in N \). In this paper we study some basic properties of classical prime subsemimodules of semimodules.

Key words: and phrases: Semimodule, Prime subsemimodule, Classical prime subsemimodule.

INTRODUCTION

The semiring and semimodule are important structures that have gained an importance in recent decades as their usefulness to many disciplines has been discovered and exploited. In spite of their similarities to the more commonly studied module structure, there are important and noteworthy differences. Subsemimodules of semimodules are different from submodules of modules in which there are several kinds of submodules. In this paper we introduce and study some concepts in semimodules which have been studied in modules.

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring we mean an algebraic system \((S, +, \cdot)\) such that \((S, +)\) and \((S, \cdot)\) are commutative semigroups, connected by \( a(b + c) = ab + ac \) for all \( a, b, c \in S \), and there exists \( 0 \in S \) such that \( r + 0 = r \) and \( r \cdot 0 = 0 \) for all \( r \in S \). Throughout this paper let \( S \) be a commutative semiring. \( S \) is called entire if for every \( a, b \in S \), \( ab = 0 \) implies that \( a = 0 \) or \( b = 0 \). A (left) semimodule \( M \) over \( S \) is a commutative additive semigroup which has a zero element, together with a mapping from \( S \times M \) into \( M \) (sending \((r, m)\) to \( rm \)) such that \( (r + s)m = rm + sm \), \( r(m + m') = rm + rm' \), \( r(sm) = (rs)m \) and \( 0m = r0_{M} = 0_{M} \) for all \( m, m' \in M \) and \( r, s \in S \). Let \( M \) be a semimodule over the semiring, \( S \) and let \( N \) be a subset of \( M \). We say that \( N \) is a subsemimodule of \( M \), or an \( S \)-subsemimodule of \( M \), precisely when \( N \) is itself an \( S \)-semimodule with respect to the operations of \( M \) (so \( 0_{M} \in N \)). The semiring \( S \) is considered to be also a semimodule over itself. In this case, the subsemimodules of \( S \) are called ideals of \( S \). Let \( M \) be a semimodule over \( S \). A subtractive subsemimodule (= \( k \)-subsemimodule) \( N \) is a subsemimodule of \( M \) such that if \( m, m' \in N \), then \( m' \in N \) (so \( \{0_{M}\} \) is a \( k \)-subsemimodule of \( M \)). We shall say that \( M \) is a distributive semimodule if the lattice of its subsemimodules is distributive, i.e., if \( (X + Y) \cap Z = (X \cap Z) + (Y \cap Z) \) for all subsemimodules \( X \), \( Y \) and \( Z \) of \( M \) (or, equivalently, \((X \cap Z) + (Y \cap Z) = (X + Y) \cap Z \) for all subsemimodules \( X \), \( Y \) and \( Z \) of \( M \)).

Behboodi and Koohi in (2004) defined a different class of submodules and called it weakly prime (classical prime). A proper submodule \( N \) of \( M \) is said to be classical prime when for \( a, b \in R \) and \( m \in M \), \( abm \in N \) implies that \( am \in N \) or \( bm \in N \). In this paper we introduce the concept of classical prime subsemimodules of a semimodule \( M \), and study some basic properties of this class of subsemimodules. Finally we endow \( Cl.Spec(M) \), the set of all classical prime subsemimodules of \( M \), with quasi-Zariski topology.

Classical Prime Subsemimodules:

Let \( M \) be a semimodule over a commutative semiring \( S \). A proper subsemimodule \( N \) of \( M \) is called classical prime if for every \( a, b \in S \), \( m \in M \), whenever \( abm \in N \), then either \( am \in N \) or \( bm \in N \). Clearly, in the case...
where $M = R$, classical prime subsemimodules coincide with prime ideals. Also every prime subsemimodule of $M$ is a classical prime subsemimodule of $S$. Here we provide an example to show that the converse is not necessarily true.

**Example 2.1:**
Assume that $S$ is an entire semiring, and $P$ is a non-zero prime ideal of $S$. In this case $Q := P \oplus 0$ is a classical prime subsemimodule of the $S$-semimodule $S \oplus S$ while it is not a prime subsemimodule. This example shows that a classical prime subsemimodule need not be prime.

Since every prime subsemimodule is classical prime, we have $\text{Spec}(M) \subseteq \text{Cl.Spec}(M)$. As it is mentioned in example 2.1, it happens sometimes that this containment is strict. We call $M$ a compatible $S$-semimodule if its classical prime subsemimodules and prime subsemimodules coincide, that is if $\text{Spec}(M) = \text{Cl.Spec}(M)$. If $S$ is a semiring, then every classical prime ideal of $S$ is a prime ideal. So, if we consider $S$ as an $S$-semimodule, it is compatible.

**Theorem 2.2:**
Let $M$ be an $S$-semimodule and $N$ a proper $K$-subsemimodule of $M$. The following statements are equivalent.

1. $N$ is a classical prime subsemimodule.
2. $abK \subseteq N$ implies that either $aK \subseteq N$ or $bK \subseteq N$ for every subsemimodule $K$ of $M$ and $a, b \in S$.
3. For every subsemimodule $K$ of $M$ not contained in $N$, $(N :_S K)$ is a prime ideal of $S$.
4. For each $m \in M \setminus N$, $(N :_S m)$ is a prime ideal of $S$.
5. $IK \subseteq N$ implies that $IK \subseteq N$ or $JK \subseteq N$ for every subsemimodule $K$ of $M$ and ideals $I, J$ of $S$.

**Proof:**

$(1) \Rightarrow (2)$ Assume that $N$ is a classical prime subsemimodule of $M$. Let $abK \subseteq N$ for some subsemimodule $K$ of $M$ and $a, b \in S$. If $aK \subseteq N$ and $bK \subseteq N$, then there exist $x, y \in K$ such that $ax \notin N$ and $by \notin N$. In this case from $abxy \in N$ we get $hx \in N$ and $ay \in N$. In this case it follows from $ab(x + y) \in K \subseteq N$ that either $ax + ay \in N$ or $bx + by \in N$. If $ax + ay \in N$, then $ax \in N$ and $ay \in N$ is a $k$-subsemimodule, a contradiction. If $bx + by \in N$, we get a contradiction in a similar way.

The other implications are clear.

**Proposition 2.3:**
Let $N$ be a proper subsemimodule of the $S$-semimodule $M$. Then $N$ is a prime subsemimodule of $M$ if and only if $N$ is primary and classical prime.

**Proof:**
If $N$ is a prime subsemimodule of $M$, then it clearly is both primary and classical prime. Conversely, assume that $N$ is a primary and classical prime subsemimodule of $M$. Let $a \in S$ and $m \in M$ be such that $am \in N$ but $m \notin N$. Then, there exists a positive integer $k$ such that $a^k \in (N :_S M)$ since $N$ is assumed to be primary. Consequently, for every $y \in M \setminus N$, $a^k \in (N :_S y)$ and $(N :_S y)$ is a prime ideal of $S$ by Theorem 2.2. Hence $a \in (N :_S y)$, that is $ay \in N$. Therefore $a \in (N :_S M)$, that is $N$ is a prime subsemimodule.

Let $M$ be a semimodule over $S$. A subsemimodule $N$ of $M$ is called a partitioning subsemimodule ($= Q_M$-subsemimodule) if there exists a non-empty subset $Q_M$ of $M$ such that

1. $S_Q = \{q \in S : q \in Q_M\}$;
2. $M = \bigcup \{q + N : q \in Q_M\}$;
3. If $q_1, q_2 \in Q_M$, then $(q_1 + N) \cap (q_2 + N) = \emptyset$ if and only if $q_1 = q_2$.

It is easy to see that if $Q_M = M$, then $\{0\}$ is a $Q_M$-subsemimodule of $M$. Let $N$ be a $Q_M$-subsemimodule of $M$. We put $M/N = \{q + N : q \in Q_M\}$. Then $M/N$ forms a commutative additive semigroup which has zero element under the binary operation $\oplus$ defined by $(q_1 + N) \oplus (q_2 + N) = q_1 + q_2 + N$ where $q \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q + N$. By the definition of $Q_M$-subsemimodule, there exists a unique $q_0 \in Q_M$
such that $0_M + N \subseteq q_0 + N$. Then $q_0 + N$ is a zero element of $M/N$. But, for every $q \in Q_M$ from (1) one obtains $0_M = 0_M q = 0_M$. Hence $q_0 = 0_M$. Now let $r \in S$ and suppose that $q_1 + N, q_2 + N \in M/N$ are such that $q_1 + N = q_2 + N$ in $M/N$. Then $q_1 = q_2$, we must have $r(q_1 + N) = r(q_2 + N)$ by the Definition of $Q_M$-subsemimodule. Hence we can unambiguously define a mapping from $S \times M/N$ into $M/N$ (sending $(r, q_1 + N)$ to $r(q_1 + N)$) and it is routine to check that this turns the commutative semigroup $M/N$ into an $S$-semimodule.

We call this $S$-semimodule the residue class semimodule or factor semimodule of $M$ modulo $N$ (Ebrahimi Atani, 2010). Let $N$ a $Q_M$-subsemimodule of $M$. It is shown in (Ebrahimi Atani, 2010) that if $L$ is a $k$-subsemimodule of $M$ with $N \subseteq L$, then $L/N = \{q + N : q \in L \cap Q_M\}$ is a $k$-subsemimodule of $M/N$. Conversely, if $L$ is a $k$-subsemimodule of $M/N$, then $L = T/N$ for some $k$-subsemimodule $T$ of $M$.

**Theorem 2.4:**

Let $M$ be an $S$-semimodule and $N$ be a $Q_M$-subsemimodule of $M$. If $K$ is a $k$-subsemimodule of $M$ such that $N \subseteq K$, then $K$ is a classical prime subsemimodule of $M$ if and only if $K/N$ is a classical prime subsemimodule of $M/N$.

**Proof:**

Assume first that $K$ is a classical prime subsemimodule of $M$. So $K \neq M$ and therefore $K/N \neq M/N$ by [2, Theorem 4]. Suppose that $ab(q + N) \in K/N$, where $a, b \in S$ and $q \in K \cap Q_M$. Then $ab(q + N) = abq + N \in K/N$, so $abq \in K$. Thus $K$ classical prime gives $aq \in K$ or $bq \in K$, so $aq \in K \cap Q_M$ or $bq \in K \cap Q_M$. Therefore $a(q + N) = aq + N \in K/N$ or $b(q + N) = bq + N \in K/N$. That is $K/N$ is a classical prime subsemimodule of $M/N$. Conversely, assume that $K/N$ is a classical prime subsemimodule of $M/N$. Then $K/N \neq M/N$ implies that $K \neq M$. If $abm \in K$ where $a, b \in S$ and $m \in M$, then there exists $q \in Q_M$ and $n \in N$ such that $m = q + n$, so $ab(q + n) = abq + abn \in K$. Since $K$ is a $k$-subsemimodule, we have $abq \in K \cap Q_M$. So $ab(q + n) = abq + N \in K/N$, therefore $aq + N \in K/N$ or $bq \in K/N$ since $K/N$ is assumed to be classical prime. This means $aq \in K$ or $bq \in K$, hence $K$ is a classical prime subsemimodule of $M$.

Let $M$ be an $S$-semimodule. An element $r \in S$ is called a zero-divisor on $M$ if $rm = 0$, for some nonzero $m \in M$. The set of all zero-divisors of $S$ on $M$ is denoted by $Zdv(S)(M)$. Let $N$ be a subsemimodule of $M$. An element $r \in S$ is said to be prime to $N$ if $rm \in N$ (with $m \in M$) implies that $m \in N$, that is, $(N \cap S) r = N$, where $(N \cap S) r = \{m \in M | rm \in N\}$. We will denote the set of all elements of $S$ that are not prime to $N$ by $S(N)$. $S(N)$ is not necessarily an ideal of $S$. A proper subsemimodule $N$ is called a primal subsemimodule of $M$ if $S(N)$ forms an ideal of $R$. If $N$ is a primal subsemimodule, then the ideal $P := S(N)$ is a prime ideal of $S$, called the adjoint prime ideal of $S$. In this case we say that $N$ is a $P$-primal subsemimodule of $M$. Now assume that $N$ is a $Q_M$-subsemimodule of $M$. Then it is easy to prove that $N$ is a $P$-primal subsemimodule of $M$ if and only if $Zdv_S(M/N)$ is a prime ideal of $S$. It is also straightforward to check that for a subsemimodule $N$ of $M$,

$$(N :_S M) \subseteq Zdv_S(M/N) = \bigcup_{m \in M} (N :_S m)$$

**Corollary 2.5:**

Let $N$ be a classical prime subsemimodule of the $S$-semimodule $M$. Then.

1. $P := (N :_S M)$ is a prime ideal of $S$. In this case we say that $N$ is a $P$-classical prime subsemimodule of $M$.
2. $(N :_S M)_{\text{max} M/N}$ is a chain of prime ideals of $S$.
3. $N$ is a primal subsemimodule of $M$.
4. For all subsemimodules $K$ and $L$ of $M$ not contained in $N$, $(N :_S K) \subseteq (N :_S L)$ or $(N :_S L) \subseteq (N :_S K)$.
Proof:

(1) It follows from Theorem 2.2.

(2) For every \( m_1, m_2 \in \mathcal{M} \setminus N \), \((N : \zeta m_1) \cap (N : \zeta m_2) \subseteq (N : \zeta (m_1 + m_2))\). If \( m_1 + m_2 \notin N \), then \((N : \zeta m_1 + m_2)\) is a prime ideal of \( S \) by Theorem 2.2, and if \( m_1 + m_2 \in N \), then \((N : \zeta m_1 + m_2) = S \). In either case it follows that \((N : \zeta m_1) \subseteq (N : \zeta m_1 + m_2)\) or \((N : \zeta m_2) \subseteq (N : \zeta m_1 + m_2)\). Therefore, \((N : \zeta m_1) = (N : \zeta m) \cap (N : \zeta m_2 + m_1) \subseteq (N : \zeta m_1)\). Consequently, \(\{(N : \zeta m)\}_{m \in \mathcal{M} \setminus N}\) is a chain of prime ideals of \( S \).

(3) It follows from (2).

(4) Assume that \( K \) and \( L \) are subsemimodules of \( M \) not contained in \( N \). We have \((N : \zeta L) \subseteq (N : \zeta K) \subseteq (N : \zeta L + K)\). By Theorem 2.2, \((N : \zeta L + K)\) is a prime ideal of \( S \) so we have \((N : \zeta L) \subseteq (N : \zeta L + K)\) or \((N : \zeta K) \subseteq (N : \zeta L + K)\), and this implies that \((N : \zeta L) = (N : \zeta L) \cap (N : \zeta L + K) \subseteq (N : \zeta K)\) or \((N : \zeta L) = (N : \zeta L) \cap (N : \zeta L + K) \subseteq (N : \zeta K)\).

Theorem 2.6:

Let \( M \) be an \( S \)-semimodule and \( N \) a proper \( \mathcal{K} \)-subsemimodule of \( M \). The following statements are equivalent:

(i) \( \bar{N} \) is a classical prime subsemimodule.

(ii) For any \( x, y \in M \), if \((N : \zeta x) \neq (N : \zeta y)\), then \( N = (N + Sx) \cap (N + Sy) \).

Proof:

(i) \( \Rightarrow \) (ii) Let \( \bar{N} \) be a classical prime subsemimodule. Assume that \( x, y \in M \) are such that \((N : \zeta x) \neq (N : \zeta y)\). Then there exists an element \( r \in (N : \zeta x) \setminus (N : \zeta y)\). In this case we have \( rx \notin N \) and \( ry \notin \bar{N} \). Therefore, \( y \in \mathcal{M} \setminus N \) and \((N : \zeta y) \) is a prime ideal of \( S \) by Theorem 2.2. We claim that \((N : \zeta y) \subseteq (N : \zeta ry)\). Clearly, \((N : \zeta y) \subseteq (N : \zeta ry)\). For the other containment, let \( a \in (N : \zeta ry)\). Then \((N : \zeta y) \subseteq (N : \zeta ry)\).

(ii) \( \Rightarrow \) (i) Assume that (ii) holds, and let \( abm \in N \) for some \( a, b \in \mathcal{M} \) and \( m \in M \). Suppose that \( am \notin N \). Then \( a \in (N : \zeta bm) \setminus (N : \zeta m) \), so \((N : \zeta bm) \neq (N : \zeta m) \). Hence, by our assumption, we have, \( N = (N + Sm) \cap (N + Shm) \). Now we have \( bm \in (N + Sm) \cap (N + Shm) = N \) and the proof is completed.

Quasi-Zariski Topology on \( \text{Cl.Spec}(M) \)

The set of all classical prime subsemimodules of the \( S \)-semimodule \( M \) is denoted by \( \text{Cl.Spec}(M) \). As an example, the zero semimodule has no classical prime subsemimodules. So there are some semimodules that have no classical prime subsemimodules. We call such semimodules classical primeless. A subsemimodule \( N \) of \( M \) is said to be classical semiprime if \( N \) is an intersection of classical prime subsemimodules of \( M \). Let \( N \) be a subsemimodule of \( M \). The classical radical of \( N \) in \( M \), denoted by \( \sqrt[\text{C}]{N} \), is defined to be the intersection of all classical prime subsemimodules of \( M \) containing \( N \). So if \( \text{Cl.Spec}(M) = \emptyset \), then \( \sqrt[\text{C}]{N} = M \), and if \( \text{Cl.Spec}(M) \neq \emptyset \), then \( \sqrt[\text{C}]{N} \) is a classical semiprime subsemimodule.

For each subsemimodule \( N \) of \( M \), let \( V(N) = \{ P \in \text{Spec}(M) \mid N \subseteq P \} \). In this case, the set \( \zeta(M) = \{ V(N) \mid N \text{ is a subsemimodule of } M \} \) contains the empty set and \( \text{Spec}(M) \), and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The \( S \)-semimodule \( M \) is said to be a

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Top semimodule if \( \zeta(M) \) is closed under finite unions. In this case \( \zeta(M) \) satisfies the axioms for the closed sets of a unique topology \( \tau(M) \) on \( Spec(M) \). The topology \( \tau(M) \) on \( Spec(M) \) is called the quasi-Zariski topology. In the remainder of this section we use a similar method to define a topology on \( Cl.Spec(M) \). For each subsemimodule \( N \) of \( M \), set

\[
V(N) = \{ P \in Cl.Spec(M) \mid N \subseteq P \}.
\]

**Proposition 3.1:**

Let \( M \) be an \( S \)-semimodule. Then

1. For each subset \( E \subseteq M \), \( V(E) = V(N) = V(\sqrt{N}) \), where \( N \) is the subsemimodule of \( M \) generated by \( E \).
2. \( V(0) = Cl.Spec(M) \), and \( V(M) = \emptyset \).
3. If \( \{ N_{i} \}_{i \in I} \) is a family of subsemimodules, then \( \bigcap_{i \in I} V(N_{i}) = V(\sum_{i \in I} N_{i}) \).
4. For every pair \( N \) and \( K \) of subsemimodules, we have \( V(N) \cup V(K) \subseteq V(N \cap K) \).

**Proof:**

(1) Assume that \( N \) is the subsemimodule of \( M \) generated by \( E \subseteq M \). Then from \( E \subseteq N \subseteq \sqrt{N} \) we have \( V(\sqrt{N}) \subseteq V(N) \subseteq V(E) \).

On the other hand, \( N \) is the smallest subsemimodule of \( M \) containing \( E \), so that if \( P \in V(E) \), then \( P \in V(N) \). Therefore \( V(E) = V(N) \). Moreover \( \sqrt{N} \) is the intersection of all classical prime subsemimodules of \( M \) containing \( N \); so \( V(N) \subseteq V(\sqrt{N}) \). Therefore \( V(E) = V(N) = V(\sqrt{N}) \).

The proof of (2)–(4) is straightforward.

Let \( M \) be an \( S \)-semimodule. Set \( \eta(M) = \{ V(N) \mid N \text{ is a subsemimodule of } M \} \) then by Proposition 3.1,

(i) \( \eta(M) \) contains the empty set and \( Cl.Spec(M) \); and (ii) \( \eta(M) \) is closed under arbitrary intersections, but it is not necessarily closed under finite unions.

**Definition 3.2:**

Let \( M \) be an \( S \)-semimodule.

1. \( M \) is said to be a \( Cl.Top \) semimodule if \( \eta(M) \) is closed under finite unions, i.e. for any subsemimodules \( N \) and \( L \) of \( M \) there exists a subsemimodule \( K \) of \( M \) such that \( V(N) \cup V(L) = V(K) \).
2. A classical prime subsemimodule \( N \) of \( M \) will be called classical extraordinary (\( Cl.extraordinary \) for short), if whenever \( K \) and \( L \) are classical semiprime subsemimodules of \( M \) with \( K \cap L \subseteq N \) then \( K \subseteq N \) or \( L \subseteq N \).

Note that if \( M \) is a \( Cl.Top \) semimodule, then \( \eta(M) \) satisfies the axioms for the closed sets of a unique topology \( \rho(M) \) on \( Cl.Spec(M) \). In this case, the topology \( \rho(M) \) on \( Cl.Spec(M) \) is called the quasi-Zariski topology. Note that every \( Cl.Primeless \) semimodules is a \( Cl.Top \) semimodule. In the next Theorem, we provide a useful tool for characterization of \( Cl.Top \) semimodules.

**Theorem 3.3:** Let \( M \) be a \( S \)-semimodule. Then, the following statements are equivalent:

(i) \( M \) is a \( Cl.Top \) semimodule.

(ii) Every classical prime subsemimodule of \( M \) is \( Cl.extraordinary \).

(iii) \( V(N) \cup V(L) = V(N \cap L) \) for any classical semiprime subsemimodules \( N \) and \( L \) of \( M \).

**Proof:**

If \( M \) is \( Cl.Primeless \), then the result is clear. So assume that \( Cl.Spec(M) \neq \emptyset \).

(i) \( \Rightarrow \) (ii) Let \( M \) be a \( Cl.Top \) module. Assume that \( P \) is a graded classical prime subsemimodule of \( M \) and that \( N, L \) are classical semiprime subsemimodules of \( M \) with \( N \cap L \subseteq P \). By the hypothesis, there
exists a subsemimodule $K$ of $M$ with $\mathcal{V}(N)\cup\mathcal{V}(L) = \mathcal{V}(K)$. As $N$ is considered to be a classical semiprime subsemimodule, we have $N = \bigcap_{i \in I} P_i$ where $\{P_i\}_{i \in I}$ is a collection of classical prime subsemimodules of $M$. For every $i \in I$, we have

$$P_i \in \mathcal{V}(N) \subseteq \mathcal{V}(K) \Rightarrow K \subseteq P_i \Rightarrow K \subseteq \bigcap_{i \in I} P_i = N$$

In a similar way we have $K \subseteq L$. So $K \subseteq N \cap L$. Now

$$\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L) \subseteq \mathcal{V}(K) = \mathcal{V}(N) \cup \mathcal{V}(L).$$

Consequently, $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(N \cap L)$. Now from $N \cap L \subseteq P$ we have $P \in \mathcal{V}(N \cap L) = \mathcal{V}(N) \cup \mathcal{V}(L)$. Hence either $P \in \mathcal{V}(N)$ or $P \in \mathcal{V}(L)$, that is either $N \subseteq P$ or $L \subseteq P$. So $P$ is Cl.extraordinary.

$(ii) \Rightarrow (iii)$  Suppose that all classical prime subsemimodules of $M$ are Cl.extraordinary. Let $N$ and $L$ be two classical semiprime subsemimodules of $M$. Clearly $\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L)$. Now assume that $P \in \mathcal{V}(N \cap L)$. Then $N \cap L \subseteq P$. Since $P$ is Cl.extraordinary, we have $N \subseteq P$ or $L \subseteq P$, that is either $P \in \mathcal{V}(N)$ or $P \in \mathcal{V}(L)$. Therefore $\mathcal{V}(N \cap L) \subseteq \mathcal{V}(N) \cup \mathcal{V}(L)$, and so $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(N \cap L)$.

$(iii) \Rightarrow (i)$  Assume that $N, L$ are two subsemimodules of $M$. We can assume that $\mathcal{V}(N)$ and $\mathcal{V}(L)$ are both nonempty, for otherwise $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(N)$ or $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(L)$. We know that $\sqrt{\mathcal{V}(N)}$ and $\sqrt{\mathcal{V}(L)}$ are both classical semiprime subsemimodules of $M$. Setting $K = \sqrt{\mathcal{V}(N)} \cap \sqrt{\mathcal{V}(L)}$ we have:

\[
\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(\sqrt{\mathcal{V}(N)}) \cup \mathcal{V}(\sqrt{\mathcal{V}(L)}) = \mathcal{V}(\sqrt{\mathcal{V}(N) \cap \sqrt{\mathcal{V}}} = \mathcal{V}(K)
\]

by $(iii)$. Hence $M$ is a Cl.Top semimodule. \(\Box\)

**Corollary 3.4:**

Every Cl.Top semimodule is a Top semimodule.

**Proof:**

Assume that $M$ is a Cl.Top semimodule. Let $P$ be a prime subsemimodule of $M$. Since every prime subsemimodule is a classical prime subsemimodule, $P$ is Cl.extraordinary by Proposition 3.3. Hence it is extraordinary. Now the result follows from (Ebrahimi Atani, S. and U. Tekir, 2011).

**Theorem 3.5:**

If $M$ is a Cl.Top $S$-semimodule, then

1. The $S$-semimodule $M/K$ is a Cl.Top semimodule for every $Q_{\text{cl}}$-subsemimodule $K$ of $M$.
2. If $\sqrt{\mathcal{V}(N)} = N$ for every subsemimodule $N$ of $M$, then $M$ is a distributive semimodule.

**Proof:**

If $M$ is classical primeless, then we are done. So assume that $\text{Cl.Spec}(M) \neq \emptyset$.

1. By Proposition 2.4, the classical prime subsemimodules of $M/K$ are just the subsemimodules $N/K$ where $N$ is a classical prime subsemimodule of $M$ with $K \subseteq N$. Consequently, any classical semiprime subsemimodule of $M/K$ is of the form $S/K$ in which $S$ is a classical semiprime subsemimodule of $M$ with $K \subseteq S$. Assume that $S_i/K$ and $S_j/K$ are two classical semiprime subsemimodules of $M/K$. Then, by Theorem 3.3, $\mathcal{V}(S_i) \cup \mathcal{V}(S_j) = \mathcal{V}(S_i \cap S_j)$ since $M$ is a Cl.Top semimodule. Thus $\mathcal{V}(S_i/K) \cup \mathcal{V}(S_j/K) = \mathcal{V}(S_i/K \cap S_j/K)$.

It follows from Theorem 3.3 that $M/K$ is a Cl.Top module.

3. For every subsemimodules $N, K$ and $L$ of $M$ we have:

\[
(K + L) \cap N = \sqrt{\mathcal{V}(K + L) \cap N}
= \bigcap \{P \mid P \in \mathcal{V}((K + L) \cap N) \}
= \bigcap \{P \mid P \in \mathcal{V}(K + L) \cup \mathcal{V}(N) \}
= \bigcap \{P \mid P \in \mathcal{V}(K) \cap \mathcal{V}(L) \cup \mathcal{V}(N) \}
= \bigcap \{P \mid P \in (\mathcal{V}(K) \cup \mathcal{V}(N)) \cap (\mathcal{V}(L) \cup \mathcal{V}(N)) \}
\]
Thus $M$ is distributive.

Let $M$ be a $\text{Cl.} \text{Top}$ $S$-semimodule and let $X = \text{Cl.} \text{Spec}(M)$. We know that any closed subset of $X$ is of the form $\mathcal{V}(N)$ for some subsemimodule $N$ of $M$. Now there is a question: what open subsets of $X$ look like. For every subset $S$ of $M$, define $X_S = X - \mathcal{V}(S)$.

In particular, if $S = \{f\}$, we denote $X_S$ be $X_f$.

**Proposition 3.6:**

The set $\{X_f \mid f \in M\}$ is a basis for the quasi-Zariski topology on $X$.

**Proof:**

Let $U$ be a non-void open subset in $X$. Then $U = X - \mathcal{V}(N)$ for some subsemimodule $N$ of $M$.

Assume that $N$ is generated by some subset $E \subseteq M$. Then we have

$$U = X - \mathcal{V}(N) = X - \mathcal{V}(\bigcup_{f \in E} f) = X - \bigcap_{f \in E} \mathcal{V}(f) = \bigcup_{f \in E} (X - \mathcal{V}(f)) = \bigcup_{f \in E} X_f$$

Therefore the set $\{X_f \mid f \in M\}$ is a basis for $X$.

**REFERENCES**


