A Reliable Algorithm For Solving Cauchy-Euler Differential Equation

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Abstract: In this paper, exact analytical solutions of Cauchy-Euler differential equation are obtained by the differential transformation method (DTM). The method is capable of reducing the size of calculations and handles linear or nonlinear, homogeneous or nonhomogeneous equations, in a direct manner. Four examples are presented to illustrate the efficiency and reliability of the method.

Keywords: Differential transformation method; Taylor's series expansion, Cauchy-Euler differential equation.

INTRODUCTION

A linear differential equation of the form

\[ a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = g(x), \tag{1} \]

Where the coefficients \( a_n, a_{n-1}, \ldots, a_0 \) are constants, is known as a Cauchy-Euler equation. The observable characteristic of this type of equation is that the degree \( k = n, n - 1, \ldots, 1, 0 \) of the monomial coefficients \( x^k \) matches the order \( k \) of the differentiation \( \frac{d^k y}{dx^k} \). The Cauchy-Euler equation is a linear equation with variable coefficients whose general solution can always be expressed in term of powers of \( x \), sines, cosines and logarithmic functions.

In this paper we present a reliable algorithm based on DTM to obtain the exact analytical solutions of the Cauchy-Euler equation.

The concept of differential transformation method was first proposed by Zhou (1986) in 1986; see (Chen, C.L., 1998) and (1998; Chen, C.L., et al., 1996; Chen, C.K., et al., 1996) and it was applied to solve linear and non-linear initial value problems in electric circuit analysis. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of differential equations. For example, it was applied to two point boundary value problems (Chen, C.L., 1998), to differential-algebraic equations (Ayaz, F., 2004), to the KdV and mKdV equations (Kangalgil, F., F. Ayaz, 2009), to the Schrödinger equations (Ravi, S.V., K. Kanth, Aruna, 2009), to fractional differential equations (Arikoglu, A., I. Ozkol, 2007) and to the Riccati differential equation (Biazar, J., M. Eslami, 2010). Jang et al., (2000) introduced the application of the concept of the differential transformation of fixed grid size to approximate solutions of linear and non-linear initial value problems. Hassan (2004) applied the differential transformation technique of fixed grid size to solve the higher-order initial value problems. The transformation method can be used to evaluate the approximating solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation. The main advantage of this method is that it can be applied directly to linear and nonlinear ODEs without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate (Chang, S.H., I.L. Chang, 2008).

The differential transformation method (DTM):

An arbitrary function \( f(x) \) can be expanded in Taylor series about a point \( x = 0 \) as
The differential transformation of \( f(x) \) is defined as

\[
F(x) = \frac{1}{k!} \left[ \frac{d^k f}{dx^k} \right]_{x=0}.
\]

Then the inverse differential transform is

\[
f(x) = \sum_{k=0}^{\infty} x^k F(k).
\]

The fundamental mathematical operations performed by differential transform method are listed in Table 1.

### Table 1: The fundamental operations of differential transformation method (DTM).

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(x) = g(x) + h(x) )</td>
<td>( Y(k) = G(k) + H(k) )</td>
</tr>
<tr>
<td>( y(x) = \alpha g(x) )</td>
<td>( Y(k) = \alpha G(k) )</td>
</tr>
<tr>
<td>( y(x) = \frac{dg(x)}{dx} )</td>
<td>( Y(k) = (k + 1)G(k + 1) )</td>
</tr>
<tr>
<td>( y(x) = \frac{d^2g(x)}{dx^2} )</td>
<td>( Y(k) = (k + 1)(k + 2)G(k + 2) )</td>
</tr>
<tr>
<td>( y(x) = \frac{d^m g(x)}{dx^m} )</td>
<td>( Y(k) = (k + 1)(k + 2)\ldots(k + m)G(k + m) )</td>
</tr>
<tr>
<td>( y(x) = 1 )</td>
<td>( Y(k) = \delta(k) )</td>
</tr>
<tr>
<td>( y(x) = x )</td>
<td>( Y(k) = \delta(k - 1) )</td>
</tr>
<tr>
<td>( y(x) = x^m )</td>
<td>( Y(k) = \delta(k - m) = \begin{cases} 1, &amp; k = m \ 0, &amp; k \neq m \end{cases} )</td>
</tr>
<tr>
<td>( y(x) = g(x)h(x) )</td>
<td>( Y(k) = \sum_{m=0}^{k} H(m)G(k - m) )</td>
</tr>
<tr>
<td>( y(x) = e^{\lambda x} )</td>
<td>( Y(k) = \frac{\lambda^k}{k!} )</td>
</tr>
<tr>
<td>( y(x) = (1 + x)^m )</td>
<td>( Y(k) = \frac{m(m - 1)\ldots(m - k + 1)}{k!} )</td>
</tr>
</tbody>
</table>

The operation properties of differential transformation:

If \( x(t), y(t) \) are two uncorrelated functions with time \( t \) and \( X(k), Y(k) \) are the transformed functions corresponding to \( x(t), y(t) \) and the basic properties are shown as follows:

If \( X(k) = D[x(t)], Y(k) = D[y(t)] \) and \( c_1 \) and \( c_2 \) are independent of \( t \) and \( k \), then

\( D[c_1 x(t) + c_2 y(t)] = c_1 X(k) + c_2 Y(k) \).

(Symbol \( D \) denoting the differential transformation process).

If \( z(t) = x(t)y(t), x(t) = D^{-1}[X(k)], y(t) = D^{-1}[Y(k)] \) and \( \otimes \) denote the convolution, then
If \( y(x) = y_1(x)y_2(x) \ldots y_{n-1}(x)y_n(x) \) then

\[
Y(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_1} Y_1(k_1)Y_2(k_2-k_1)\ldots Y_{n-1}(k_{n-1}-k_{n-2})Y_n(k-k_{n-1}).
\]

Applications:
To illustrate the effectiveness of the DTM we shall consider four test examples for the Cauchy-Euler equation.

Example:
First, we present an analytical solution to the simple Cauchy-Euler equation:

\[
x^2 y'' + 3xy' = 0, \quad y(1) = 0, \quad y'(1) = 4.
\]

The analytical solution of the above problem is given by,

\[
y(x) = 2 - 2x^{-2}.
\]

We first should overcome the difficulties encountered by the singularity at \( x = 0 \). To achieve this goal, we use the transformation

\[
z = \ln x, \quad x = e^z,
\]

so that

\[
\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz}
\]

\[
\frac{d^2y}{dx^2} = \frac{1}{x^2} \frac{d^2y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz}.
\]

Using (7) and (8) into (5) gives

\[
\frac{d^2y}{dz^2} + 2 \frac{dy}{dz} = 0,
\]

With the conditions

\[
y(z = 0) = 0, \quad y'(z = 0) = 4.
\]

By using the fundamental operations of differential transformation method in Table 1, we obtained the following recurrence relation for equation (9):

\[
Y(k+2) = -\frac{1}{(k+1)(k+2)}\left[2(k+1)Y(k+1)\right].
\]
From the initial condition (10) we have \( Y(0) = 0 \), \( Y(1) = 4 \) and from equation (11) we have

\[
\begin{align*}
Y(2) &= 4, \\
Y(3) &= 8, \\
Y(4) &= -\frac{4}{3}, \\
Y(5) &= \frac{8}{15}, \\
&\vdots
\end{align*}
\]

Therefore, the closed form of the solution can be easily written as

\[
y(z) = \sum_{k=0}^{\infty} Y(k) z^k = 4z - 4z^2 + \frac{8}{3}z^3 - \frac{4}{3}z^4 + \frac{8}{15}z^5 - \ldots
\]

(12)

\[
= 2 - 2e^{-2z}.
\]

(13)

Recall that \( x = e^z \) then

\[
y(x) = 2 - 2x^2.
\]

(14)

Which is the exact solution.

**Example:**

Now, we consider the following classical Cauchy-Euler equation:

\[
x^2y'' + xy' + y = 0, \quad y(1) = 1, \quad y'(1) = 2.
\]

(15)

The analytical solution of the above problem is given by,

\[
y(x) = \cos(\ln x) + 2\sin(\ln x).
\]

(16)

To overcome the difficulties encountered by the singularity at \( x = 0 \), we use the transformation

\[
z = \ln x, \quad x = e^z,
\]

(17)

so that

\[
\begin{align*}
\frac{dy}{dx} &= \frac{1}{x} \frac{dy}{dz}, \\
\frac{d^2y}{dx^2} &= \frac{1}{x^2} \frac{d^2y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz}.
\end{align*}
\]

(18)

Using (17) and (18) into (15) gives

\[
\frac{d^2y}{dz^2} + y = 0.
\]

(19)

With the conditions
By using the fundamental operations of differential transformation method in Table 1, we obtained the following recurrence relation for equation (19):

\[ Y(k + 2) = -\frac{1}{(k + 1)(k + 2)}[Y(k)] \]  

(21)

From the initial condition (20) we have \( Y(0) = 1, \ Y(1) = 2 \) and from equation (21) we have

\[ Y(2) = -\frac{1}{2}, \]
\[ Y(3) = -\frac{1}{3}, \]
\[ Y(4) = \frac{1}{24}, \]
\[ Y(5) = \frac{1}{60}, \]
\[ \vdots \]

Therefore, the closed form of the solution can be easily written as

\[ y(z) = \sum_{k=0}^{\infty} Y(k) z^k = 1 + 2z^2 - \frac{1}{3}z^3 + \frac{1}{24}z^4 + \frac{1}{60}z^5 - \ldots \]  

(22)

\[ = \cos(z) + 2 \sin(z). \]  

(23)

Recall that \( x = \ln z \) then

\[ y(x) = \cos(\ln x) + 2 \sin(\ln x). \]  

(24)

Which is the exact solution.

**Example:**
Solve the second order Cauchy-Euler equation:

\[ x^2 y'' - 2xy' + 2y = 0, \]

(25)

Subject to the initial conditions

\[ y(1) = 2, \quad y'(1) = 3. \]  

(26)

With the exact solution

\[ y(x) = x + x^2. \]  

(27)

To overcome the difficulties encountered by the singularity at \( x = 0 \), we use the transformation

\[ z = \ln x, \quad x = e^z, \]

(28)

So that
\[
\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz}
\]
\[
\frac{d^2y}{dx^2} = \frac{1}{x^2} \frac{d^2y}{dz^2} + \frac{1}{x^2} \frac{dy}{dz}.
\] (29)

Using (28) and (29) into (25) gives
\[
\frac{d^2y}{dz^2} - 3 \frac{dy}{dz} + 2y = 0,
\] (30)
with the conditions
\[
y(z = 0) = 2, \quad y'(z = 0) = 3.
\] (31)

Taking the differential transform of (30), leads to:
\[
Y(k + 2) = \frac{1}{(k + 1)(k + 2)} [3(k + 1)Y(k + 1) - 2Y(k)]
\] (32)

From the initial conditions given by Eq. (31) we have
\[
Y(0) = 2, \quad Y(1) = 3.
\] (33) (34)

Substituting Eq. (32) into Eqs. (33) and (34) and by recursive method, the results are listed as follows
\[
Y(2) = \frac{5}{2},
\]
\[
Y(3) = \frac{3}{2},
\]
\[
Y(4) = \frac{17}{24},
\]
\[
Y(5) = \frac{11}{40},
\]
\[
\vdots
\]

Therefore, the closed form of the solution can be easily written as
\[
y(z) = \sum_{k=0}^{\infty} Y(k)z^k = 2 + 3z + \frac{5}{2}z^2 + \frac{3}{2}z^3 + \frac{17}{24}z^4 + \frac{11}{40}z^5 + \ldots
\] (35)
\[
e^z + e^{2z}.
\] (36)

Recall that \( z = \ln x \) then
\[
y(x) = x + x^2.
\] (37)

Which is the exact solution.

**Example:**
Consider the following Cauchy-Euler differential equation:
\[ xy'' + y' = x, \quad (38) \]

Subject to the initial conditions
\[ y(1) = 1, \quad y'(1) = -\frac{1}{2}. \quad (39) \]

With the exact solution
\[ y(x) = \frac{3}{4} + \frac{1}{4} x^2 - \ln x. \quad (40) \]

In order to change Eq. (38) to standard Cauchy-Euler form, we multiply (38) by \( x \) to get
\[ x^2 y'' + xy' = x^2, \quad (41) \]

Use the transformation
\[ z = \ln x, \quad x = e^z, \quad (42) \]

So that
\[ \frac{dy}{dz} = \frac{dy}{dx} \frac{dx}{dz} \]

This gives
\[ \frac{d^2 y}{dz^2} - e^{2z} = 0, \quad (44) \]

With the conditions
\[ y(z = 0) = 1, \quad y'(z = 0) = -\frac{1}{2}. \quad (45) \]

Taking the differential transform of (44), leads to:
\[ Y(k + 2) = \frac{1}{(k + 1)(k + 2)} \left[ \frac{2^k}{k!} \right] \quad (46) \]

From the initial conditions given by Eq. (45) we have
\[ Y(0) = 1, \quad (47) \]
\[ Y(1) = -\frac{1}{2}. \quad (48) \]

Substituting Eq. (46) into Eqs. (47) and (48) and by recursive method, the results are listed as follows
\[ Y(2) = \frac{1}{2}, \]
Therefore, the closed form of the solution can be easily written as

\[ y(z) = \sum_{k=0}^{\infty} Y(k)z^k = 1 - \frac{1}{2}z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{6}z^4 + \frac{1}{15}z^5 + \ldots \]  

(49)

By \( z = \ln x, \quad x = e^z \) we have

\[ y(x) = \frac{3}{4} + \frac{1}{4}x^2 - \ln x. \]

(51)

Which is the exact solution.

**Conclusions:**

In this paper, the differential transform method is proposed for solving Cauchy-Euler differential equation. Four examples are given to illustrate the validity and accuracy of the method. The differential transformation method was used in a direct way without using linearization, perturbation or restrictive assumptions. This method, unlike most numerical techniques, provides a closed-form solution. It is shown that differential transformation method is a very fast convergent, precise and cost efficient tool for solving the Cauchy-Euler differential equation.

**REFERENCES**


